Analytic Evaluation of Volatility Forecasts*

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Abstract

The development of estimation and forecasting procedures using empirically realistic continuous-time stochastic volatility models is severely hampered by the lack of closed-form expressions for the transition densities of the observed returns. In response to this, Andersen, Bollerslev, Diebold and Labys (2002) have recently advocated modeling and forecasting the (latent) integrated volatility of primary import from a pricing perspective based on simple reduced form time series models for the observable realized volatilities, constructed from the summation of high-frequency squared returns. Building on the eigenfunction stochastic volatility class of models introduced by Meddahi (2001), we present analytical expressions for the loss in forecast efficiency associated with this easy-to-implement procedure as a function of the sampling frequency of the returns underlying the realized volatility measures. On numerically quantifying this efficiency loss for such popular continuous-time models as GARCH, multi-factor affine, and log-normal diffusions, we find that the realized volatility reduced form procedures perform remarkably well in comparison to the optimal (non-feasible) forecasts conditional on the full sample path realization of the latent instantaneous volatility process.

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1 Introduction

Continuous-time stochastic volatility models figure prominently in modern asset pricing theories. At the same time, empirical analysis of such models are generally complicated by intractable expressions for the transition density of the observed discrete-time returns. Even though a burst of research activity over the last few years has allowed for important new advances (see, e.g., the surveys in Aït-Sahalia, Hansen and Scheinkman, 2002, Gallant and Tauchen, 2002, and Johannes and Polson, 2002), the discrete-time (G)ARCH class of models remains the workhorse for modeling and forecasting time-varying volatility in situations of practical import. Recent advances in econometric methodology and richer data sources hold the promise of a paradigm shift.

In particular, from the perspectives of asset pricing and risk management, interests typically center on (forecasts for) the *integrated volatility* as opposed to the point-in-time volatility which often serves as a (latent) state variable in the formulation of continuous-time models. Moreover, from a statistical perspective, the integrated volatility provides a direct measure of the discrete-time return variability appropriately defined (see, e.g., the discussion in Andersen, Bollerslev, Diebold and Labys, 2002, henceforth ABDL, and Andersen, Bollerslev and Diebold, 2002). These observations, along with the increased availability of continuously recorded intraday prices (ultra high-frequency data in the terminology of Engle, 2000), have spurred much novel research into the measurement, modeling, and forecasting of integrated volatility based on discretely-sampled *realized volatilities* constructed from the summation of finely sampled squared high-frequency returns (e.g., Andersen and Bollerslev, 1998, ABDL, 2001, Barndorff-Nielsen and Shephard, 2001, 2002a,b, Comte and Renault, 1998, and Meddahi, 2002a).

The empirical results of ABDL (2002) are particularly intriguing, suggesting that relatively simply discrete-time ARMA based forecasts for the realized volatilities compare admirably to forecasts based on the standard set of volatility models employed in the academic literature and most commonly used by practitioners. Of course, from a formal statistical perspective these easy-to-compute reduced form time series forecasts for the observed realized volatilities invariably entail a loss in efficiency relative to the optimal, but generally infeasible, forecasts for the latent integrated volatilities based on the true underlying continuous-time model.

We provide explicit analytical expressions for the expected future integrated volatility for the most popular stochastic volatility diffusion models employed in the literature, including GARCH, multi-factor affine, and log-normal diffusions. We obtain these results by exploiting the general eigenfunction stochastic volatility model class introduced by Meddahi (2001). By conditioning the expectations on the full sample path realization of the latent volatility process[1] as well as the coarser information set consisting of only the lagged realized volatilities constructed from the high-frequency returns over fixed-length time intervals, our results allow for a direct assessment of the tradeoff between modeling complexity, sampling frequency, and forecast accuracy. As such, it also directly quantifies the loss associated with simple feasible procedures relative to the optimal, but infeasible ones.[2]

Following Andersen and Bollerslev (1998) and ABDL (2002) among others, we focus our forecast comparisons on the value of the coefficient of multiple correlation in the ex-post regressions of the (latent) integrated volatility of interests on the forecasts obtained from the different volatility modeling procedures.[3] On numerically quantifying this loss for empirically realistic sampling frequencies for several specific models recently reported in the literature, we find that the simple discrete-time autoregressive models for the realized volatilities perform remarkably well compared to the fully efficient (non-feasible) continuous-time model forecasts conditional on the full sample-path realization of the latent volatility process. Hence, our results lend additional theoretical support to the use of simple empirical reduced form modeling and forecasting procedures based on the observable realized volatilities in situations of practical import.

The plan for the rest of the paper is as follows. The next section formally defines the notions of integrated and realized volatility within the class of continuous-time stochastic volatility models. We also briefly review the arguments for focusing on forecasts of integrated volatility based on projections involving realized volatility.

[1] Throughout the paper, we identify the path realization of the volatility process and the path realization of the state variable driving the volatility process. We will be more specific when the difference between these two paths is important for forecasting purposes.

[2] This type of analysis parallels previous studies related to the predictability of mean asset returns; see, e.g. the discussion in Campbell, Garcia, Meddahi and Sentana (2002).

[3] No universally acceptable loss function exists for the evaluation of non-linear model forecasts; see, e.g., the discussion in Andersen, Bollerslev and Lange (1999) and Christoffersen and Diebold (2000). The particular loss function used here is directly inspired by the earlier contributions of Mincer and Zarnowitz (1969), and we will refer to the corresponding regression as such; see also the discussion in Chong and Hendry (1986).

Section 3 introduces the eigenfunction stochastic volatility class of models underlying our theoretical derivations, and also sets out the three specific models that form the basis for our numerical calculations. Section 4 presents analytical expressions for the optimal (non-feasible) one- and multi-step-ahead forecasts for the integrated volatility conditional on the full sample path realization of the latent spot volatility process, along with the less efficient (still non-feasible) forecasts conditional on the coarser information set consisting of "only" the lagged integrated volatilities. Section 5 in turn presents the (feasible) forecasts for the future integrated volatilities conditional on the past observable realized volatilities. This section also quantifies the corresponding loss in efficiency for each of the three illustrative candidate models as a function of the sampling frequency of the returns used in the construction of the realized volatilities. Section 6 concludes. All proofs are relegated to a technical Appendix.

2 Integrated and Realized Volatility

We focus on a single asset traded in a liquid financial market. Assuming the sample-path of the corresponding logarithmic price process, $\{p_t, 0 \leq t\}$, to be continuous,[4] the class of continuous-time stochastic volatility models traditionally employed in the finance literature is conveniently expressed in terms of the following generic stochastic differential equation (sde),

$$dp_t = \mu_t dt + \sigma_t dW_t \tag{2.1}$$

where W_t denotes a standard Brownian motion, and the drift term μ_t is (locally) predictable and of finite variation.[5] The point-in-time, or spot, volatility process $\{\sigma_t, 0 \leq t\}$ measures the instantaneous strength of the price variability expressed per unit-of-time.

Following standard practice we assume that the sample path of the σ_t process is also continuous. Generally, σ_t and W_t may be contemporaneously correlated so that a so-called leverage style effect is allowed. However, the (asymptotic) distributional result discussed in this section is only known to be true under the assumption that

[4]The discussion in this section explicitly rules out discontinuities in the price process. However, our new theoretical results based on the eigenfunction stochastic volatility class of models could fairly easily be extended to allow for jumps. We briefly allude to this possibility in Section 3.

[5]The drift, μ_t, may generally depend explicitly on both p_t and σ_t. However, we suppressed all the arguments for notational simplicity.

$d\sigma_t$ and dW_t are uncorrelated (no leverage effect). Likewise, the results concerning realized volatility in Section 5 precludes leverage effects. In contrast, the new theoretical results in Sections 3 and 4 explicitly allow for a non-zero (instantaneous) correlation, and we conjecture that the key results on realized volatility in Section 5 will be (approximately) valid in this case as well. Likewise, to facilitate the exposition, we explicitly exclude jump processes although many of the results remain valid for empirically relevant jump specifications.

The sde in equation (2.1) greatly facilitates arbitrage-based pricing arguments. However, as emphasized by Andersen, Bollerslev and Diebold (2002), practical return calculations and volatility measurements are invariably restricted to discrete time intervals. In particular, focusing on the unit time interval, the one-period continuously compounded return corresponding to (2.1) is formally given by,

$$r_t \equiv p_t - p_{t-1} = \int_{t-1}^{t} \mu_u du + \int_{t-1}^{t} \sigma_u dW_u. \tag{2.2}$$

Hence, with no leverage effect and conditional on the sample-path realizations of the drift and volatility processes, $\{\mu_u, t-1 \leq u \leq t\}$ and $\{\sigma_u, t-1 \leq u \leq t\}$, the one-period returns will be Gaussian with conditional mean equal to the first integral on the right-hand-side of equation (2.2), while the conditional variance equals the *integrated volatility*,

$$IV_t \equiv \int_{t-1}^{t} \sigma_u^2 du. \tag{2.3}$$

The integrated volatility therefore affords a natural measure of the (ex-post) return variability, as recently highlighted in independent work by Andersen and Bollerslev (1998), Comte and Renault (1998) and Barndorff-Nielsen and Shephard (2001). The integrated volatility also plays a key role in the stochastic volatility option pricing literature. In particular, ignoring the variation in the conditional mean, Hull and White (1987) show that option prices are uniquely determined by the expected future integrated volatility (see also Garcia, Lewis and Renault, 2001).

Of course, integrated volatility is not directly observable. This has spurred the development of several new statistical procedures for modeling and forecasting the (latent) integrated volatility based on specific parametric models within the general diffusion class of models in equation (2.1) (see, e.g., Gallant, Hsu and Tauchen, 1999, and Barndorff-Nielsen and Shephard, 2001, and the references therein). While these procedures allow for the construction of asymptotically optimal forecasts under appropriate conditions, they are generally not robust to misspecifications of the underlying continuous-time model and also quite complicated to implement.

Alternatively, consider the so-called *realized volatility* defined by the summation of intra-period squared returns,

$$RV_t(h) \equiv \sum_{i=1}^{1/h} r_{t-1+ih}^{(h)2}, \tag{2.4}$$

where the h-period return is given by $r_t^{(h)} = p_t - p_{t-h}$, and $1/h$ is a positive integer. By the theory of quadratic variation, $RV_t(h)$ converges uniformly in probability to IV_t as $h \to 0$, thus allowing for increasingly more accurate non-parametric measurements of integrated volatility as the sampling frequency of the underlying intra-period returns increases.

If we further assume that the realized and integrated volatility measures are square integrable, the asymptotic unbiasedness of $RV_t(h)$ for IV_t, implies that forecasts for IV_{t+j}, $j \geq 1$, based on the projection of $RV_{t+j}(h)$ on any time t information set, will also be (asymptotically) unbiased and optimal in a Mean-Square-Error (MSE) sense relative to that particular information set.[6] In particular, restricting the information set to the lagged realized volatilities only, as proposed by ABDL (2002), conveniently circumvents the complications associated with the use of latent variable procedures in the construction of the integrated volatility forecasts. Of course, doing so also entails a loss in forecast efficiency relative to the optimal (non-feasible) forecasts for IV_{t+j} conditional on the full sample path realization of the instantaneous price and (latent) spot volatility processes. Quite remarkably, however, as we show below, this loss in efficiency is typically fairly small. We next turn to a discussion of the eigenfunction stochastic volatility class of models used in our formal derivation of this important practical result.

[6] This result holds generally subject to a uniform integrability condition ensuring convergence in expectation of the uniformly consistent realized volatility measure. As such, it includes cases in which the continuous sample path assumption for spot volatility is violated; see, e.g., the discussion in Andersen, Bollerslev and Diebold (2002) and Barndorff-Nielsen and Shephard (2002b). Hoffman-Jørgensen (1994), sections 3.22-3.25, provides a formal discussion of the necessary uniform integrability conditions on the underlying price process to ensure convergence in expectation. See also Billingsley (1986), Exercise 21.21, for the identical result.

3 Eigenfunction Stochastic Volatility Models

This section reviews the main properties of the Eigenfunction Stochastic Volatility (ESV) models introduced in Meddahi (2001) and provides a discussion of the specific parametric models considered in our numerical calculations. The ESV class of models includes most continuous-time stochastic volatility models analyzed in the existing literature. Meanwhile, the formulation in terms of orthogonal eigenfunctions provides a particular convenient and elegant framework for the derivation of explicit analytical expressions for volatility forecasts.

3.1 General Theory

The generic stochastic volatility model in equation (2.1) is only restricted by the requirement that the point-in-time, or spot, volatility process, σ_t, be non-negative. Most popular stochastic volatility models in the existing literature are based on the additional assumptions that the volatility process is driven by a single (latent) state variable. In the context of the ESV class of models, the corresponding one-factor representation takes the form,

$$dp_t = \mu_t dt + \sigma_t \; [\sqrt{1-\rho^2} dW_t^{(1)} + \rho dW_t^{(2)}], \tag{3.1}$$

where $W_t^{(1)}$ and $W_t^{(2)}$ denote two independent standard Brownian Motions, and the instantaneous volatility is related to the latent state variable,

$$df_t = \mu(f_t)dt + \sigma(f_t)dW_t^{(2)}, \tag{3.2}$$

by the functional relationship,

$$\sigma_t^2 = \sum_{i=0}^{p} a_i P_i(f_t), \tag{3.3}$$

where the integer p may be infinite, the a_i coefficients are real numbers, the $P_i(f_t)$'s denote the eigenfunctions of the infinitesimal generator, $\mathcal{A}$, associated with f_t, and the normalizations $P_0(f_t) = 1$ and $Var[P_i(f_t)] = 1$ for $i \neq 0$ are imposed for notational simplicity.[7] Moreover, the stationary process for $\{f_t\}$ is assumed

[7]The infinitesimal generator, $\mathcal{A}$, associated with f_t is formally defined by

$$\mathcal{A}\phi(f_t) \equiv \mu(f_t)\phi'(f_t) + \frac{\sigma^2(f_t)}{2}\phi''(f_t),$$

to be time reversible, and therefore the set of eigenvalues of the infinitesimal generator associated with f_t is real; see Meddahi (2001) for further discussion of these additional regularity conditions for the general ESV class of models.

The expression for σ_t^2 in equation (3.3) may appear somewhat arbitrary. Importantly, however, any square-integrable function $g(f_t)$ can be written as a linear combination of the eigenfunctions associated with f_t, i.e.,

$$g(f_t) = \sum_{i=0}^{\infty} a_i P_i(f_t), \tag{3.4}$$

where $a_i = E[g(f_t)P_i(f_t)]$ and $\sum_{i=0}^{\infty} a_i^2 = E[g(f_t)^2] < \infty$, so that $g(f_t)$ is the limit in mean-square of $\sum_{i=0}^{p} a_i P_i(f_t)$ for p going to infinity. As such, the ESV structure encompasses the popular GARCH diffusion model (Nelson, 1990) corresponding to $\sigma_t^2 = f_t$ and $f_t = k[\theta - f_t]dt + \sigma f_t dW_t^{(2)}$, as well as the log-normal model (Hull and White, 1987; Wiggins, 1987) obtained by setting $\sigma_t^2 = \exp(f_t)$ and $f_t = k[\theta - f_t]dt + \sigma dW_t^{(2)}$, and the one-factor square-root model of Heston (1993) defined by $\sigma_t^2 = f_t$ and $f_t = k[\theta - f_t]dt + \sigma\sqrt{f_t}dW_t^{(2)}$.

The power of the ESV representations of these and other continuous time stochastic volatility models essentially stems from the following two properties. First, the eigenfunctions associated with different eigenvalues are orthogonal and any nonconstant eigenfunction is centered at zero (for $i, j > 0$ and $i \neq j$):

$$E[P_i(f_t)P_j(f_t)] = 0 \quad \text{and} \quad E[P_i(f_t)] = 0. \tag{3.5}$$

These features, of course, underlie the result noted after equation (3.4) that $\sum_{i=0}^{\infty} a_i^2 = E[g(f_t)^2]$. Second, the eigenfunctions are first order autoregressive processes (in general heteroskedastic):

$$\forall l > 0,\ E[P_i(f_{t+l}) \mid f_\tau, \tau \leq t] = \exp(-\lambda_i l) P_i(f_t). \tag{3.6}$$

Given the structure of the ESV model and the Markovian nature of the joint process (p_t, f_t), conditional expectations of any transformation of this variable, including the variance, therefore only depend on the expectations of the eigenfunctions. The

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for any square-integrable and twice differentiable function, $\phi(f_t)$. The corresponding eigenfunctions, $P_i(f_t)$, and eigenvalues, $(-\lambda_i)$, satisfy

$$\mathcal{A}P_i(f_t) = -\lambda_i P_i(f_t).$$

For a more detailed discussion of the properties of infinitesimal generators see e.g., Hansen and Scheinkman (1995) and Aït-Sahalia, Hansen and Scheinkman (2002).

orthogonality of the eigenfunctions coupled with the simple first-order autoregressive dynamics in turn render such calculations straightforward.

The ESV representations discussed above are based on a single (latent) state variable. Meanwhile, several recent studies, including Alizadeh, Brandt and Diebold (2002), Bollerslev and Zhou (2002), Engle and Lee (1999), Gallant, Hsu and Tauchen (1999), and Harvey, Ruiz and Shephard (1994) among others, have argued for the empirical relevance of allowing for multiple volatility factors. Without loss of generality, consider the two-factor case. Following Meddahi (2001), let $f_{1,t}$ and $f_{2,t}$ denote two independent stochastic processes characterized by,

$$df_{j,t} = \mu_j(f_{j,t})dt + \sigma_j(f_{j,t})dW_t^{(j+1)}, \quad j = 1, 2, \tag{3.7}$$

with the eigenfunctions and eigenvalues of the corresponding infinitesimal generator denoted by $P_{j,i}(f_{j,t})$ and $(-\lambda_{j,i})$, $j = 1, 2$, respectively. The variance process for the general two-factor ESV model is then defined by

$$\sigma_t^2 = \sum_{0 \le i,j \le p} a_{i,j} P_{1,i}(f_{1,t}) P_{2,j}(f_{2,t}), \tag{3.8}$$

where in analogy to the one-factor case, the $a_{i,j}$ coefficients are square summable. The properties of the eigenfunctions in (3.5) and (3.6) similarly hold true for the functions $P_{i,j}^{(2)}(f_t) \equiv P_{1,i}(f_{1,t})P_{2,j}(f_{2,t})$, where $f_t \equiv (f_{1,t}, f_{2,t})^\top$. Hence, the $P_{i,j}^{(2)}(f_t)$'s are simply the eigenfunctions associated with the bivariate state variable $(f_{1,t}, f_{2,t})^\top$, and the same calculations outlined above goes through in this case.[8]

3.2 Specific Examples

The numerical analysis presented in subsequent sections is based on three specific models, namely a GARCH diffusion model, a two-factor affine model, and a log-normal diffusion. Meddahi (2001) shows how each of these models may be represented as an ESV model by explicitly solving for the corresponding eigenfunctions. The following subsections provide a brief summary of these results, along with the actual parameter values used in the numerical calculations.

[8]See Chen, Hansen and Scheinkman (2000) for a general approach to eigenfunction modeling in the multivariate case.

3.2.1 Model M1 - GARCH Diffusion

The instantaneous volatility in the GARCH diffusion model is defined by the process,

$$d\sigma_t^2 = k(\theta - \sigma_t^2)dt + \sigma\sigma_t^2 dW_t^{(2)}.$$

This model was first introduced by Wong (1964) and later popularized by Nelson (1990). The model is readily expressed as an ESV model by defining the state variable,

$$df_t = k(\theta - f_t)dt + \sigma f_t dW_t^{(2)},$$

and the function $g(x) = x$. Assuming that the variance of σ_t^2 is finite, it is possible to show that

$$\sigma_t^2 = a_0 + a_1 P_1(f_t),$$

where $a_0 = \theta$, $a_1 = \theta\sqrt{\psi/(1-\psi)}$, $\psi = \sigma^2/2k$, and the first and only eigenfunction for f_t is affine,

$$P_1(x) = \frac{\sqrt{1-\psi}}{\theta\sqrt{\psi}}(x - \theta).$$

As discussed above, this representation of the process greatly facilities any expected variance and/or volatility forecast calculations. Note also that the second moment of the variance σ_t^2 is assured to be finite for ψ less than one. In the numerical calculations reported on here we rely on the parameters from Andersen and Bollerslev (1998) as implied from the (weak) daily GARCH(1,1) model estimates for the DM/dollar from 1987 through 1992 using the temporal aggregation results of Drost and Nijman (1993) and Drost and Werker (1996). In particular, $k = 0.035$, $\theta = 0.636$, and $\psi = 0.296$. These parameter values were also used in the studies by Andersen, Bollerslev and Lange (1999) and Andreou and Ghysels (2002).

3.2.2 Model M2 - Two-Factor Affine

The instantaneous variance in the two-factor affine model is given by,

$$\sigma_t^2 = \sigma_{1,t}^2 + \sigma_{2,t}^2, \qquad d\sigma_{j,t}^2 = k_j(\theta_j - \sigma_{j,t}^2)dt + \eta_j\sigma_{j,t}dW_t^{(j+1)}, \qquad j = 1, 2.$$

Following Meddahi (2001), this model may be cast in the form of an ESV model by defining the state variables,

$$df_{j,t} = k_j(\alpha_j + 1 - f_{j,t})dt + \sqrt{2k_j}\sqrt{f_{j,t}}dW_t^{(j+1)}, \qquad j = 1, 2,$$

where $\alpha_j = (2k_j\theta_j/\eta_j^2) - 1$, and the $f_{j,t}$'s are related to the $\sigma_{j,t}$'s by the functional relationship,

$$f_{j,t} = \frac{2k_j}{\eta_j^2}\sigma_{j,t}^2, \qquad j = 1, 2.$$

In particular, it is possible to show that the eigenfunctions associated with $f_{j,t}$ are given by the Laguerre polynomials $L_i^{(\alpha_j)}(f_{j,t})$, $i = 0, 1, ...,$ with corresponding eigenvalues $\lambda_{j,i} = k_j i$. Moreover,

$$\sigma_{j,t}^2 = \tilde{a}_{j,0}L_0^{(\alpha_j)}(f_{j,t}) + \tilde{a}_{j,1}L_1^{(\alpha_j)}(f_{j,t})$$

with $\tilde{a}_{j,0} = \theta_j$ and $\tilde{a}_{j,1} = -\sqrt{\theta_j}\eta_j/\sqrt{2k_j}$, so that by equation 3.8),

$$\sigma_t^2 = a_{0,0} + a_{1,0}L_1^{(\alpha_1)}(f_{1,t}) + a_{0,1}L_1^{(\alpha_2)}(f_{2,t}),$$

where $a_{0,0} = \tilde{a}_{1,0} + \tilde{a}_{2,0}$, $a_{1,0} = \tilde{a}_{1,1}$ and $a_{0,1} = \tilde{a}_{2,1}$. The actual numerical results for the two-factor model are based on the parameter estimates reported in Bollerslev and Zhou (2002) obtained by matching the sample moments of the daily realized volatilities constructed from high-frequency five-minute DM/dollar returns spanning 1986 through 1996 to the corresponding population moments for the integrated volatility. The resulting values are, $k_1 = 0.5708$, $\theta_1 = 0.3257$, $\eta_1 = 0.2286$, $k_2 = 0.0757$, $\theta_2 = 0.1786$, $\eta_2 = 0.1096$, implying the existence of a very volatile first factor, along with a much more slowly mean reverting second factor.

3.2.3 Model M3 - Log-Normal Diffusion

Our last numerical example is based on the log-normal diffusion model,

$$d\log(\sigma_t^2) = k[\theta - \log(\sigma_t^2)]dt + \sigma dW_t^{(2)}.$$

Again, following Meddahi (2001), this model may be expressed in the form of an ESV model by defining the state variable,

$$df_t = -kf_t dt + \sqrt{2k}\ dW_t^{(2)},$$

related to σ_t by the functional relationship,

$$f_t = \frac{\sqrt{2k}}{\sigma}(\log\sigma_t^2 - \theta).$$

The eigenfunctions associated with this Ornstein-Uhlenbeck process for f_t are given by the usual Hermite polynomials $H_i(f_t)$, $i = 0, 1, ...$, with corresponding eigenvalues $\lambda_i = ki$. Hence,

$$\sigma_t^2 = \sum_{i=0}^{\infty} a_i H_i(f_t),$$

where the a_i coefficients take the form,

$$a_i = \exp(\theta + \frac{\sigma^2}{4k}) \frac{(\sigma/\sqrt{2k})^i}{\sqrt{i!}}.$$

Our numerical illustrations for this model rely on the EMM-based parameter estimates for the daily S&P500 returns spanning 1953 through 1996 reported in Andersen, Benzoni and Lund (2002), where we restrict the estimated correlation between $dW_t^{(1)}$ and $dW_t^{(2)}$ related to the leverage effect to be zero. In particular, $k = 0.0136$, $\theta = -0.8382$, $\sigma = 0.1148$. Finally, the summation in (3.3) is truncated at $p = 100$.

4 Ideal Integrated Volatility Forecasts

This section provides analytic expressions for the basic dependency structure of integrated volatility across the full range of ESV diffusion models. Building on these findings, we go on to characterize the extent of the predictability of integrated volatility for any member of this important class of asset price processes. The predictability is obviously dependent on the assumed information set. We present results ranging from the ideal case of knowing the current (latent)volatility state, through knowing the spot volatility, to observing the past sequence of integrated volatility only. All such information sets are unattainable in practice, as they contain variables that are not observed, but rather must be estimated or extracted from discrete return data. Nonetheless, they serve as important benchmarks that establish the maximal predictability and reveal, step-by-step, how much forecast power is lost as we condition on successively less informative, but empirically more readily approximated, variables. The results based on conditioning the forecasts on past integrated volatility sets the stage for the analysis of the feasible integrated volatility forecasts based on realized volatility measures extracted directly from observed high-frequency, intraday data in the following section.

4.1 Optimal One-Period-Ahead Forecasts

Our first proposition characterizes the basic first and second moment properties of the spot and integrated volatility processes within the ESV diffusion class (the proofs of all the propositions are given in the technical Appendix). The results are all new at this level of generality, but some expressions are clearly related to the abstract characterization of the second moment properties of integrated volatility in Barndorff-Nielsen and Shephard (2002a). Moreover, these authors also provide some concrete results for special cases of the ESV model as well as the non-Gaussian Ornstein-Uhlenbeck model.

Proposition 4.1 *For any ESV diffusion model, as defined in Section 3.1 (with potential non-zero drift or leverage effects), and any integer $n \geq 1$, we have*

$$E[\sigma_t^2] = E[IV_{t+n}] = a_0, \tag{4.1}$$

$$E[\sigma_{t+n}^2 \mid p_\tau, f_\tau, \tau \leq t] = a_0 + \sum_{i=1}^{p} a_i \exp(-\lambda_i n) P_i(f_t), \tag{4.2}$$

$$E[IV_{t+n} \mid p_\tau, f_\tau, \tau \leq t] = a_0 + \sum_{i=1}^{p} a_i \exp(-\lambda_i (n-1)) \frac{[1-\exp(-\lambda_i)]}{\lambda_i} P_i(f_t), \tag{4.3}$$

$$Var[\sigma_t^2] = \sum_{i=1}^{p} a_i^2, \tag{4.4}$$

$$Var[IV_t] = 2\sum_{i=1}^{p} \frac{a_i^2}{\lambda_i^2}[\exp(-\lambda_i) + \lambda_i - 1], \tag{4.5}$$

$$Cov(IV_{t+n}, \sigma_t^2) = \sum_{i=1}^{p} a_i^2 \exp(-\lambda_i (n-1)) \frac{[1-\exp(-\lambda_i)]}{\lambda_i}, \tag{4.6}$$

$$Cov(IV_{t+n}, IV_t) = \sum_{i=1}^{p} a_i^2 \exp(-\lambda_i (n-1)) \frac{[1-\exp(-\lambda_i)]^2}{\lambda_i^2}, \tag{4.7}$$

$$Cov(\sigma_{t+n}^2, \sigma_t^2) = \sum_{i=1}^{p} a_i^2 \exp(-\lambda_i n) \tag{4.8}$$

One set of novel implications is given by the following corollary to Proposition 4.1. It provides rather intuitive, yet useful, ranking of the second moment expressions for the different latent volatility notions.

Proposition 4.2 *Under the assumptions of Proposition 4.1, and for any integer* $n \geq 1$*, we have*

$$Cov(\sigma^2_{t+n}, \sigma^2_t) \leq Cov(IV_{t+n}, IV_t) \leq Cov(IV_{t+n}, \sigma^2_t) \leq Var[IV_t] \leq Var[\sigma^2_t]. \quad (4.9)$$

The inequalities are most readily comprehended by recalling the timing between spot and integrated volatility and recognizing that integrated volatility is a smoothed version of the spot volatility process. The first inequality, for example, reflects the fact that the spot volatilities are separated by n periods whereas the gap between the intervals over which the integrated volatilities, IV_{t+n} and IV_t, are measured is only $n-1$ periods. The second inequality suggests that the spot volatility at the interval end is more informative about future integrated volatility than the smoothed (average) spot volatility over the corresponding interval. Such a conclusion appears natural in the single eigenfunction case, given the Markov structure of ESV models, but is less obvious for the multiple eigenfunction case where affine functions of spot volatility cannot provide a sufficient statistic for the volatility state vector, f_t. The final inequalities are hardly surprising, but have important implications. The variability of either volatility measure exceeds the covariability measures and the smoothed integrated volatility measure is less variable than the spot volatility. The latter finding implies that the comparatively high covariance between spot volatility and future integrated volatility may be due to (excess) variability of spot volatility rather than superior correlation between spot and future (integrated) volatility. Hence, as discussed extensively below, it is not clear in general whether conditioning on the spot or the integrated volatility will allow for the more efficient forecast.

We are now in position to assess the optimal integrated volatility forecasts generated by different information sets. We adopt the standard expected quadratic loss function, implying that optimal forecasts equal the conditional expectation of integrated volatility given the available information. The universally best forecast is based on the full history of the log-price and volatility path, denoted by the sigma algebra, $\sigma(p_\tau, f_\tau, \tau \leq t)$. As discussed above, given the Markov structure of the ESV models, knowledge of the state vector, f_t, is sufficient for the construction of the optimal predictor. The coefficient of multiple correlation, or R^2, from the

Mincer-Zarnowitz type regression of future integrated volatility on the corresponding (conditional expectation) forecasts (and a constant) serves as a popular and useful summary measure of forecast performance. Moreover, it is consistent with the emphasis on quadratic loss. For notational convenience, we focus our analysis on the one-period-ahead integrated volatility, IV_{t+1}, but similar results are readily available for the n-period-ahead measure, IV_{t+n}.

Proposition 4.1 and the orthogonality of the eigenfunctions, indicated in (3.5), imply that the "explained variation" from regressing IV_{t+1} onto $E[IV_{t+1} \mid p_\tau, f_\tau, \tau \leq t]$ (and a constant), denoted $R^2(IV_{t+1}, Best)$, is given by

$$R^2(IV_{t+1}, Best) = \frac{1}{Var[IV_t]} \sum_{i=1}^{p} a_i^2 \frac{[1 - \exp(-\lambda_i)]^2}{\lambda_i^2}, \tag{4.10}$$

with $Var[IV_t]$ determined by (4.5).

Alternatively, consider forecasts based on the current spot volatility. In one-factor ESV models, there is no difference between the conditional expectation of integrated volatility given either spot volatility or the volatility state vector, so the two forecasts coincide. In multi-factor models, however, the spot volatility is not a sufficient statistic for the volatility state vector, with the latter being more informative. Moreover, the process (p_t, σ_t) is not Markovian, and, hence, the associated optimal forecast - the conditional expectation of IV_{t+1} given $\sigma(p_\tau, \sigma_\tau^2, \tau \leq t)$ - depends in general on the entire path of σ_t^2 and is not available in closed form.

Consequently, we next consider a simple forecast that depends linearly on σ_t^2. Obviously, the best affine forecast of IV_{t+1} is given by the corresponding (population) regression coefficients on σ_t^2 and a constant.

Using (4.6), it follows readily that the R^2 of the corresponding Mincer-Zarnowitz regression, denoted $R^2(IV_{t+1}, \sigma_t^2)$, may be expressed as,

$$R^2(IV_{t+1}, \sigma_t^2) = \frac{1}{Var[IV_t] Var[\sigma_t^2]} \left(\sum_{i=1}^{p} a_i^2 \frac{[1 - \exp(-\lambda_i)]}{\lambda_i} \right)^2 \tag{4.11}$$

where $Var[IV_t]$ and $Var[\sigma_t^2]$ are given by (4.5) and (4.4). In the univariate factor case, if the variance is solely a function of a single (non-constant) eigenfunction, this forecast still coincides with the "best". However, when the variance depends on multiple eigenfunctions, the forecast will differ from the "best", even in the case of a single factor. In the specific examples discussed in Section 3.2 above, we have one case (the GARCH diffusion) where the forecasts coincide (single factor, one eigenfunction), one where they differ in a univariate factor model (the lognormal diffusion, where the variance depends on multiple eigenfunctions), and one where the spot volatility forecast is necessarily inferior to the "best"(the two-factors affine diffusion).

Forecasts based on integrated volatility are of particular interest as they, in practice, may be approximated by the corresponding realized volatility obtained from high-frequency data. Current (and past) integrated volatility will generally not provide a sufficient statistic for the volatility state vector so the optimal forecasts are not available in closed form and we again restrict attention to forecasts generated by affine functions of the integrated volatility. In particular, on using (4.7) it follows that the R^2 from the regression of IV_{t+1} on IV_t and a constant, denoted $R^2(IV_{t+1}, IV_t)$, takes the form

$$R^2(IV_{t+1}, IV_t) = \frac{\left(\sum_{i=1}^{p} a_i^2 [1 - \exp(-\lambda_i)]^2 / \lambda_i^2\right)^2}{Var[IV_t]^2} \tag{4.12}$$

where $Var[IV_t]$ is again given by (4.5).

Obviously, the optimal forecast for the one-period integrated volatility will, by construction dominate in terms of the population R^2 from the associated Mincer-Zarnowitz regressions. More interesting is the relative performance of the forecasts in (4.11) and (4.12). Proposition 4.2 allows us to readily explore this issue.

Proposition 4.3 *Under the assumptions of Proposition 4.1, and if there is a unique eigenfunction in (3.3), we have*

$$R^2(IV_{t+1}, IV_t) \leq R^2(IV_{t+1}, \sigma_t^2). \tag{4.13}$$

Otherwise, no general ranking of $R^2(IV_{t+1}, IV_t)$ relative to $R^2(IV_{t+1}, \sigma_t^2)$ is feasible.

The proof (in the appendix) provides a direct assessment of the model features that account for the relative performance of these forecasts. In summary, low persistence of the eigenfunctions hurts the relative performance of integrated volatility based forecasts, whereas the main culprit behind poor performance of spot volatility based predictors is a large discrepancy in persistence across the eigenfunctions (and associated high variability of the spot volatility process). Interestingly, in spite of the higher covariance between the spot volatility and the future (integrated) volatility, it is generally not the case that the corresponding forecasts dominate those generated by conditioning on the integrated volatility. In fact, this is only assured when there is a single eigenfunction - in which case the spot volatility based predictor coincides with the optimal one.[9] In this situation, the integrated volatility forecasts should ideally be based on (an estimate of) current spot volatility or (an estimate of) a current integrated volatility measure covering as short an intraday interval as possible. However, practical difficulties in obtaining precise intraday-horizon volatility estimates limit the applicability of this insight.

Indeed, from a practical perspective the more relevant question is how *much* predictive power is lost for the different forecasts and how one may alleviate the loss in forecast accuracy if only historical integrated volatility based predictors are feasible. We return to the first question in our numerical comparisons in Section 4.4. One approach for addressing the second question involves the inclusion of additional (lagged) explanatory variables, as discussed formally in the following section.

4.2 Forecasts Based on Multiple Explanatory Variables

Since the optimal forecast of IV_{t+1} conditional on the history of spot volatility generally depends on the entire volatility path, it is natural to extend the expression for the best affine forecast of IV_{t+1} given σ_t^2 to include a fixed, finite number of lagged spot volatilities (and a constant) as regressors. Similarly, it is relevant to consider the best affine forecast of IV_{t+1} given IV_t and its lagged values.

For this purpose, it is convenient first to introduce some notation. For a covariance-stationary random variable (y_τ, z_t) and an integer l, we let $C(y_\tau, z_t, l)$ denote the $(l+1)$ vector defined by

$$C(y_\tau, z_t, l) = (Cov[y_\tau, z_t], Cov[y_\tau, z_{t-1}], ..., Cov[y_\tau, z_{t-l}])^\top. \tag{4.14}$$

Moreover, let $M(z_t, l)$ denote the $(l+1) \times (l+1)$ matrix whose (i, j)'th component is given by

$$M(z_t, l)[i, j] = Cov[z_t, z_{t+i-j}]. \tag{4.15}$$

The R^2 from the regression of IV_{t+1} onto a constant and $(\sigma_t^2, \sigma_{t-1}^2, ..., \sigma_{t-l}^2)$, $l \geq 0$,

[9] As noted above, the GARCH diffusion model falls in this category.

denoted $R^2(IV_{t+1}, \sigma_t^2, l)$, may then be succinctly expressed as [10]

$$R^2(IV_{t+1}, \sigma_t^2, l) = C(IV_{t+1}, \sigma_t^2, l)^\top (M(\sigma_t^2, l))^{-1} \; C(IV_{t+1}, \sigma_t^2, l)/Var[IV_t]. \quad (4.16)$$

Similarly, the R^2 of the regression of IV_{t+1} onto a constant and $(IV_t, IV_{t-1}, ..., IV_{t-l})$, $l \geq 0$, denoted $R^2(IV_{t+1}, IV_t, l)$, is given by

$$R^2(IV_{t+1}, IV_t, l) = C(IV_{t+1}, IV_t, l)^\top (M(IV_t, l))^{-1} \; C(IV_{t+1}, IV_t, l)/Var[IV_t]. \quad (4.17)$$

The final type of forecasts we consider is based on ARMA type representations of integrated volatility. Barndorff-Nielsen and Shephard (2002a) note that autoregressive specifications for spot volatility induce an ARMA structure for the integrated volatility process. Meddahi (2002b) shows that integrated volatility is an ARMA(1,1) process (respectively ARMA(2,2)) process if spot volatility depends on a single eigenfunction (respectively two eigenfunctions), and also provides closed-form expressions for all the ARMA parameters. Based on the ARMA representations, it is shown in the Appendix that the (population) R^2 of the corresponding regression, denoted $R^2(IV_{t+1}, ARMA)$, takes the form

$$R^2(IV_{t+1}, ARMA) = 1 - \frac{C_1}{Var[IV_t]}, \quad (4.18)$$

where C_1 refers to the variance of the innovation process in the ARMA representation of the integrated volatility. The exact definition of C_1 is likewise relegated to the Appendix.

The results above provide a detailed characterization of the properties of the various one-period-ahead volatility forecasts. We next turn to a discussion of the corresponding multi-period-ahead forecasts.

[10] Recall that the R^2 from a regression with multiple regressors, i.e., $y_t = c + x_t\beta + \eta_t$ where x_t denotes a vector of explanatory variables, is simply given by

$$R^2 = \frac{Cov(y,x) \; (Var[x])^{-1} \; Cov(x,y)}{Var[y]}.$$

4.3 Multi-Step-Ahead Forecasts

The natural "realized" benchmark for the multi-step-ahead forecasts at horizon n is given by the corresponding integrated volatility, $IV_{t+1:t+n}$,

$$IV_{t+1:t+n} = \sum_{i=1}^{n} IV_{t+i}. \tag{4.19}$$

The following proposition generalizes Proposition 4.1 to the multi-period horizon.

Proposition 4.4 *Under the assumptions of Proposition 4.1, and for integers $n \geq 1$ and $l \geq 0$*

$$E[IV_{t+1:t+n}] = na_0, \tag{4.20}$$

$$E[IV_{t+1:t+n} \mid p_\tau, f_\tau, \tau \leq t] = na_0 + \sum_{i=1}^{p} a_i \frac{[1 - \exp(-\lambda_i n)]}{\lambda_i} P_i(f_t), \tag{4.21}$$

$$Cov(IV_{t+1:t+n}, \sigma_{t-l}^2) = \sum_{i=1}^{p} a_i^2 \frac{[1 - \exp(-\lambda_i n)]}{\lambda_i} \exp(-\lambda_i l), \tag{4.22}$$

$$Cov(IV_{t+1:t+n}, IV_{t-l}) = \sum_{i=1}^{p} a_i^2 \frac{[1 - \exp(-\lambda_i)][1 - \exp(-\lambda_i n)]}{\lambda_i^2} \exp(-\lambda_i l), \tag{4.23}$$

$$Var[IV_{t+1:t+n}] = 2 \sum_{i=1}^{p} \frac{a_i^2}{\lambda_i^2} [\exp(-\lambda_i n) + \lambda_i n - 1]. \tag{4.24}$$

It is now straightforward to derive the R^2 of the corresponding Mincer-Zarnowitz regressions. In particular, in analogy to the results for the one-period-ahead case, it follows that

$$R^2(IV_{t+1:t+n}, Best) = \frac{1}{Var[IV_{t+1:t+n}]} \sum_{i=1}^{p} a_i^2 \frac{[1 - \exp(-\lambda_i n)]^2}{\lambda_i^2}, \tag{4.25}$$

$$R^2(IV_{t+1:t+n}, \sigma_t^2) = \frac{1}{Var[IV_{t+1:t+n}] Var[\sigma_t^2]} \left(\sum_{i=1}^{p} a_i^2 \frac{[1 - \exp(-\lambda_i n)]}{\lambda_i} \right)^2, \tag{4.26}$$

$$R^2(IV_{t+1:t+n}, IV_t) = \frac{\left(\sum_{i=1}^{p} a_i^2[1-\exp(-\lambda_i)][1-\exp(-\lambda_i n)]/\lambda_i^2\right)^2}{Var[IV_{t+1:t+n}]Var[IV_t]}, \tag{4.27}$$

$$R^2(IV_{t+1:t+n}, \sigma_t^2, l) = \frac{C(IV_{t+1:t+n}, \sigma_t^2, l)^\top M(\sigma_t^2, l)^{-1}\ C(IV_{t+1:t+n}, \sigma_t^2, l)}{Var[IV_{t+1:t+n}]}, \tag{4.28}$$

where $Var[IV_{t+1:t+n}]$, $Var[IV_t]$ and $Var[\sigma_t^2]$ are given respectively in (4.24), (4.5) and (4.4). Finally, the R^2 of the regression of IV_{t+1} onto a constant and $(IV_t, IV_{t-1}, ..., IV_{t-l})$, $l \geq 0$, denoted by $R^2(IV_{t+1:t+n}, IV_t, l)$, may be expressed as

$$R^2(IV_{t+1:t+n}, IV_t, l) = \frac{C(IV_{t+1:t+n}, IV_t, l)^\top M(IV_t, l)^{-1}\ C(IV_{t+1:t+n}, IV_t, l)}{Var[IV_{t+1:t+n}]}. \tag{4.29}$$

The next section offers some illustrative numerical calculations for each of the different (non- feasible) R^2 measures discussed above.

4.4 Quantifying the Forecast Performance

The specific diffusions underlying the numerical benchmark calculations reported below are detailed in Section 3.2. We begin with the one-period-ahead performance measures.

4.4.1 One-Period-Ahead Forecast Performance

The population R^2's of the Mincer-Zarnowitz regressions for the forecast of the one-period-ahead integrated volatilities are given in Table 1. First, it is noteworthy that when only one volatility factor is employed in the ESV diffusion, then the factor tends to be strongly serially correlated, leading to a high degree of predictability for the integrated volatility (models M1 and M3), irrespective of the volatility measure in the information set. Second, if another factor is brought into the model, it will typically allow for a more volatile and less persistent factor in the volatility dynamics (model M2). This lowers the fundamental persistence and predictability of the volatility process and renders the integrated volatility measures more noisy indicators of current (spot) volatility. Hence, the corresponding forecasts based on

$IV_{t-j},\ j \geq 0$, become less accurate relative to the forecasts that exploit more current information. Third, there is little evidence that the addition of lagged variables to the information set has any practical impact on forecast performance.

4.4.2 Multi-Step-Ahead Forecast Performance

The R^2 values for the multi-step-ahead forecasts for the same three models are given in Table 2. The results are reported for forecasts covering one week ($n = 5$), two weeks ($n = 10$), and about one month ($n = 20$).

For each of the three models, a large fraction of the variability of the (integrated) volatility process remains predictable, even at the monthly horizon, although the proportion now varies substantially across the models. For models M1 and M3, the loss of explanatory power associated with the construction of forecasts from spot or integrated volatility rather than the true volatility state is still limited. Of course, for model M1 the use of spot volatility is equivalent to the use of the true volatility state. As a result, there is limited scope for improvement through the addition of lagged variables in the information set, or the use of the theoretically warranted ARMA(1,1) structure for model M1. For model M2, however, there is now an appreciable deterioration in performance as we move from full information to spot volatility, and then further on to integrated volatility. The use of additional lagged variables and the theoretically motivated ARMA(2,2) model for integrated volatility now also produces a small, but non-negligible improvement.

Overall, our investigation suggests that models with a single persistent volatility factor are relatively insensitive to the choice of variables in the information set. All natural forecast procedures do well and capture a large degree of the theoretical predictability. In contrast, there is clearly some loss in predictive power when the model contains a second, less persistent volatility factor. Moreover, for such models one may obtain non-trivial gains to the forecast power by expanding the information set to include several lags of the integrated volatility through a simple AR model or, better, a theoretically motivated ARMA structure.

5 Volatility Forecasts based on Realized Volatility

None of the forecasts discussed in the previous section are actually feasible, as the true volatility state vector, the spot volatility and the integrated volatility are all latent, when only discretely sampled price data are available. The variable amongst these that may most readily be approximated with reasonably good precision from observed data is the integrated volatility. In particular, as discussed in Section 2, the observable realized volatility consistently approximates the (latent) integrated volatility for increasingly finer sampled returns. Of course, any practical application necessarily relies on realized volatility constructed from finitely sampled asset prices, and as such inevitably embodies a measurement error vis-à-vis the corresponding integrated volatility. It is consequently important to assess the magnitude of this measurement error and the associated loss in forecast efficiency. This section addresses these issues analytically.

5.1 Theoretical Relationships

In order to assess the loss of precision in the forecast evaluation regressions, we explore the relation between integrated and realized volatility in more detail. Throughout this section, we preclude drift and leverage effects. In this setting, as shown in Barndorff-Nielsen and Shephard (2002a), and also emphasized by Andersen, Bollerslev and Diebold (2002) and Meddahi (2002a), the measurement error, $U_t(h) \equiv RV_t(h) - IV_t$, is mean-zero, serially uncorrelated, and orthogonal to the IV_t process (i.e., $Cov(U_t(h), IV_{t-i}) = 0$ for all $i \in \mathbf{Z}$). As a consequence,

$$Var[RV_t(h)] = Var[IV_t] + Var[U_t(h)], \tag{5.1}$$

and

$$Var[RV_{t+1:t+n}(h)] = Var[IV_{t+1:t+n}] + nVar[U_t(h)], \tag{5.2}$$

while

$$Cov[RV_t(h), RV_{t-i}(h)] = Cov[IV_t, IV_{t-i}] = Cov[RV_t(h), IV_{t-i}] \tag{5.3}$$

where $i \neq 0$ and $Cov[IV_t, IV_{t-i}]$ is given by (4.7).

Moreover, within the context of the ESV class of models, it follows from the results in Meddahi (2002a) that[11]

$$Var[U_t(h)] = \frac{4}{h}\left(\frac{a_0^2 h^2}{2} + \sum_{i=1}^{p}\frac{a_i^2}{\lambda_i^2}[\exp(-\lambda_i h) - 1 + \lambda_i h]\right). \tag{5.4}$$

The R^2's for the Mincer-Zarnowitz regressions involving the realized volatility are now readily derived from the corresponding R^2's for the integrated volatility. The next proposition collects the general results.

Proposition 5.1 *For any ESV diffusion model without drift and leverage effects, and with the realized volatility as the only regressor (apart from a constant), we have*

$$\begin{aligned} R^2(\cdot, RV_t(h)) &= R^2(\cdot, IV_t)\frac{Var[IV_t]}{Var[RV_t(h)]} \\ &= R^2(\cdot, RV_t(h))\frac{Var[IV_t]}{Var[IV_t] + Var[U_t(h)]}. \end{aligned} \tag{5.5}$$

With realized volatility as the dependent variable, any set of regressors, and for any integer $n \geq 1$,

$$\begin{aligned} R^2(RV_{t+1:t+n}(h), \cdot) &= R^2(IV_{t+1:t+n}, \cdot)\frac{Var[IV_{t+1:t+n}]}{Var[RV_{t+1:t+n}(h)]} \\ &= R^2(IV_{t+1:t+n}, \cdot)\frac{Var[IV_{t+1:t+n}]}{Var[IV_{t+1:t+n}] + nVar[U_t(h)]}. \end{aligned} \tag{5.6}$$

As a simple implication of this proposition, it follows that the R^2's associated with the one-period realized volatility forecast evaluation regressions are always lower than the infeasible ones determined in Proposition 4.3,

$$R^2(RV_{t+1}, RV_t) \leq R^2(IV_{t+1}, RV_t) = R^2(RV_{t+1}, IV_t) \leq R^2(IV_{t+1}, IV_t). \tag{5.7}$$

[11] The same formula has previously been derived by Barndorff-Nielsen and Shephard (2002a) under the more restrictive assumption that the spot variance is a finite linear combination of autoregressive and independent processes (corresponding to the CEV and positive Ornstein-Uhlenbeck processes). The result derived here coincides with this earlier formula in the case of a unique eigenfunction in (3.3), but otherwise is more general. Similarly, expressions corresponding to the formulas in (4.1), (4.4), (4.5), (4.7), and (4.8) have previously been established by Barndorff-Nielsen and Shephard (2002a) in their more restrictive setting, while Barndorff-Nielsen and Shephard (2002c, Chapter 7) give the second equality in (5.3).

This documents the intuitive result that the use of a (feasible) realized volatility proxy in place of the (latent) integrated volatility systematically lowers the predictive power, irrespective of whether the proxy is inserted as a regressor, regressand, or both. The main issue is, of course, how serious the loss in forecast efficiency will be in empirically realistic situations.

5.2 Illustrations based on Specific Models

5.2.1 Forecasts from Past Realized Volatility

To quantify the efficiency loss that is likely to occur in practice, Tables 3 and 4 report the population R^2's for the regressions of future integrated volatility on lagged values of realized volatility for the three specific ESV models considered previously. The sampling frequency used for the realized volatility measures correspond to 5-minute returns for a 24-hour trading day ($h = 288$), an 8-hour trading day ($h = 96$), and 30-minute returns for a 24-hour trading day ($h = 48$).

Table 3 provides an indication of the feasible predictability of one-period-ahead integrated volatility. There is a noticeable drop compared to Table 1, but a very large proportion of integrated volatility remains predictable. The predictability also improves markedly as we move from realized volatility measures constructed using 48 to 288 intraday observations. Moreover, it is evident that the use of additional lagged variables in the information set is helpful only for the more imprecise realized volatility measures based on 48 intraday price observations.

Table 4 considers the predictability of integrated volatility over longer horizons. For models M1 and M3, the conclusions mirror those for Table 3 discussed above. For model M2, it is increasingly obvious that more frequent sampling of the intraday returns is beneficial. Likewise, for the scenarios with lower predictability - long horizons and relatively infrequent sampling of intraday returns - it is more important to include additional explanatory variables in the formation of the volatility forecasts.

Of course, the R^2's reported above cannot be mimicked by actual data, since the left-hand-side variable of interest - integrated volatility - is not observable. Feasible regressions must rely on, e.g., ex-post realized volatility measures as a proxy for the realization of future integrated volatility. Since the use of such a proxy will bias the observable predictability downward, it is important to recognize the size of the potential bias. The relevant population R^2 from such feasible regressions may be derived from (5.6). Moreover, combining the above result with (5.5) allows for the derivation of the fully feasible regression R^2's based on realized volatility proxies for integrated volatility, both as regressor and regressand. Tables 5 and 6 report on the amount of predictability associated with such feasible regressions.

Compared to Table 3, Table 5 reveals a significant loss of predictive power, even for models M1 and M3. Of course, this is purely artificial as it is induced solely by the measurement error in the integrated volatility proxy. Nonetheless, the R^2's still reveal a large amount of verifiable predictability in the integrated (realized) volatility process. As before, we also find that the impact of measurement errors is mitigated when more frequent sampling is undertaken.[12]

Table 6 extends the results in Table 5 to longer forecast horizons. The general conclusions are reinforced, but one interesting difference from Tables 3 and 4 becomes apparent. It is now possible for the five-period-ahead forecasts to display a higher degree of predictability than the one-period-ahead forecasts, as evidenced by the results for models M1 and M3 with a sampling frequency of $h = 48$. This occurs because the realization of integrated volatility is approximated more accurately over five periods as opposed to just one period, relative to the true variability in the latent integrated volatility. Thus, the decline in fundamental predictability associated with a longer horizon is more than offset by the relatively smaller measurement error in the dependent variable.[13] This provides a vivid illustration of the importance of recognizing the downward bias in the true predictability induced by the need to rely on an observable proxy for the ex-post integrated volatility realizations. At the same time, it is clear that the downward bias is almost non-existent for the longer 20-period horizon, where the R^2 figures in Table 6 are very close to those in Table 4 across all models and sampling frequencies.

[12] Barndorff-Nielsen and Shephard (2002a) provide complementary one-period-ahead forecast of integrated volatility based on realized volatilities. Their (model-based) approach uses the state-space representation of the integrated volatility (combined with the Kalman filter) when the spot variance depends on autoregressive and independent factors, as in models M1 and M2. Indeed, for these two models, their one-period-ahead forecasts correspond exactly to our forecasts based on the ARMA representation of the realized volatility provided in Table 5.

[13] Formally, in (5.6) the last equation will have the fundamental predictability, $R^2(IV_{t+1:t+n}, \cdot)$, decline with n, but the ratio $Var[IV_{t+1:t+n}]/Var[RV_{t+1:t+n}(h)]$ will increase with n, and for lower values of n this may actually raise the observed R^2.

5.2.2 Forecasts from Past Daily Squared Returns

To highlight the improved "signal-to-noise ratio" achieved by employing the realized volatility measures based on high-frequency intraday data rather than the traditional approach that only relies on daily data, we finally consider the results related to the forecasts of integrated and realized volatility based on past daily squared returns, i.e., $h = 1$.

The first panel of Table 7 demonstrates that it is critical to employ long lags of daily squared returns in order to predict future (integrated) volatility with any accuracy. Consistent with the general weak GARCH principle of Drost and Werker (1996) and Meddahi (2002b), it is evident how effective a simple (recursive) GARCH structure is in parsimoniously capturing the information in the lagged squared daily returns. However, even in the best of circumstances, a comparison of the upper panel in Table 7 with Tables 3 and 4 reveals that the forecast efficiency is severely curtailed by restricting the information set to the history of daily squared returns rather than the past realized volatilities constructed from the high-frequency data.

The lower panels of Table 7 provide corresponding evidence for feasible volatility forecast regressions where the integrated volatility regressor is replaced by (feasible) realized volatility approximations computed from sampling frequencies ranging from 288 intradaily observations (5 minute returns covering 24 hours) to daily data. The use of daily squared returns as a one-step-ahead volatility proxy is representative of much of the empirically oriented volatility literature over the last decade. The use of cumulative squared daily returns as a volatility measure over longer weekly, monthly and quarterly horizons has been emphasized by French, Schwert and Stambaugh (1987) and Schwert (1989, 1990), among others.

For all scenarios in the lower panels of Table 7, we inevitably find an even lower degree of predictability than implied by the corresponding integrated volatility regressions. This is an immediate consequence of equations (5.5) and (5.6). Nonetheless, for $h = 1/288$ or $h = 1/96$ we have sufficiently good approximations to integrated volatility that the qualitative results are very similar to those in the upper panel, and even for $h = 1/48$, the loss in observed forecast power is limited. As such, this reinforces the conclusion from above: the (observable) forecast efficiency is severely curtailed if one uses only the past daily returns in forecasting the future volatility.

The bottom panel of Table 7 further documents the extraordinarily poor coherence between future daily squared returns and the (near) optimal forecasts constructed from past daily squared returns. Again, for the case where the realized volatility proxy works very poorly, the R^2's actually increase with the forecast horizon - now often dramatically so - illustrating the significance of the measurement error in a single day's squared return relative to the corresponding volatility. It is evident across all forecast horizons that the true predictability of integrated volatility is wildly underestimated by the R^2 from these feasible regressions based on volatility measures constructed from daily data. This is consistent with the findings from the large literature trying to evaluate the performance of alternative volatility forecasts by studying the R^2 from the associated Mincer-Zarnowitz regressions using daily squared returns as the ex-post volatility measure. Such studies invariably find the relative performance to be unstable and to differ across both asset classes and time periods. This is what one should expect if the dependent variable of interest - realized ex-post (integrated) volatility - is measured with a large degree of imprecision. Even for long daily samples, the findings are largely random. Only by moving towards more meaningful ex-post realized volatility measures for integrated volatility will it be possible to assess forecast performance with any degree of reliability. This is, of course, exactly the point advocated in Andersen and Bollerslev (1998). In fact, the current exposition for predictability of volatility based on daily squared returns may be seen as an analytic extension to a much broader range of models of the simulation-based investigation of the continuous-time GARCH model in Andersen and Bollerslev (1998) and Andersen, Bollerslev and Lange (1999).[14]

6 Conclusion

This paper develops a direct analytic approach to the construction and assessment of volatility forecasts for continuous-time diffusion models within the broad ESV class of models. This class incorporates the most popular volatility diffusion models in current use, and may be calibrated to account well for the major empirical features of asset return volatility. The results provide theoretical upper bounds for the degree of predictability based on optimal (infeasible) forecasts along with direct measures

[14]Specifically, the entry in Table 7, Panel 1, M1, row "GARCH", corresponds directly to the entry $m = \infty$ (infinitely frequent sampling) for DM-dollar in Table 4 of Andersen and Bollerslev (1998). The minor discrepancy between the numbers is due to the presence of small simulation errors in Andersen and Bollerslev (1998).

of the loss in forecast efficiency associated with less precise, but more practical (feasible) reduced-form realized volatility based procedures. As such, our results should serve as an important theoretical foundation and inspiration for the further development of new and improved easy-to-implement empirical volatility forecasting procedures guided by proper (optimal) benchmark comparisons. The insights obtained from empirical comparisons of options implied volatilities may likewise be improved by properly accounting for the volatility error. The techniques developed here could also be used in more effectively calibrating the type of continuous-time models routinely employed in modern asset pricing theories. We leave further theoretical and empirical work along these lines for future research.

Appendix

We start out with a couple of lemmas that we will need later on.

Lemma A.1: *We have*

$$E\left[\int_0^h P_i(f_u)du \int_0^h P_j(f_u)du\right] = \delta_{ij}\frac{2}{\lambda_i^2}[\exp(-\lambda_i h) + \lambda_i h - 1] \qquad \text{(A.1)}$$

where $\delta_{ij} = 1$ *if* $i = j$ *and* $\delta_{ij} = 0$ *if* $i \neq j$. *Moreover, for* $n \geq 1$, *we have:*

$$E\left[\int_0^h P_i(f_u)du \int_{nh}^{(n+1)h} P_j(f_u)du\right] = \delta_{ij}\exp(-\lambda_j(n-1)h)\frac{[1-\exp(-\lambda_i h)]^2}{\lambda_i^2}. \qquad \text{(A.2)}$$

Proof of Lemma A.1: By using Ito's Lemma, we have

$$\begin{aligned}
&E\left[\left(\int_0^h P_i(f_u)du\right)\left(\int_0^h P_j(f_u)du\right)\right] \\
&= E\left[\int_0^h P_i(f_u)\left(\int_0^u P_j(f_s)ds\right)du\right] + E\left[\int_0^h P_i(f_u)\left(\int_0^u P_j(f_s)ds\right)du\right] \\
&= \int_0^h\left(\int_0^u E[P_i(f_u)P_j(f_s)]ds\right)du + \int_0^h\left(\int_0^u E[P_j(f_u)P_i(f_s)]ds\right)du \\
&= \int_0^h\left(\int_0^u \exp(-\lambda_i(u-s))E[P_i(f_s)P_j(f_s)]ds\right)du \\
&+ \int_0^h\left(\int_0^u \exp(-\lambda_j(u-s))E[P_j(f_s)P_i(f_s)]ds\right)du \\
&= 2\delta_{ij}\int_0^h\left(\int_0^u \exp(-\lambda_i(u-s))ds\right)du = \delta_{ij}\frac{2}{\lambda_i^2}[\exp(-\lambda_i h) + \lambda_i h - 1],
\end{aligned}$$

i.e., (A.1). Besides, we get (A.2) as follows:

$$\begin{aligned}
E\left[\int_0^h P_i(f_u)du \int_{nh}^{(n+1)h} P_j(f_u)du\right] &= E\left[\int_0^h P_i(f_u)du \int_{nh}^{(n+1)h} E[P_j(f_u) \mid f_\tau, \tau \leq h]du\right] \\
&= E\left[\int_0^h P_i(f_u)du \int_{nh}^{(n+1)h} \exp(-\lambda_j(u-h))P_j(f_h)du\right] \\
&= \exp(-\lambda_j(n-1)h)\frac{1-\exp(-\lambda_i h)}{\lambda_i}\int_0^h E[P_i(f_u)P_j(f_h)]du \\
&= \delta_{ij}\exp(-\lambda_j(n-1)h)\frac{1-\exp(-\lambda_i h)}{\lambda_i}\int_0^h \exp(-\lambda_i(h-u))du \\
&= \delta_{ij}\exp(-\lambda_j(n-1)h)\frac{[1-\exp(-\lambda_i h)]^2}{\lambda_i^2}. \blacksquare
\end{aligned}$$

Lemma A.2: $\mathbf{R}^2$ of multi-step forecasts of an ARMA(1,1) or ARMA(2,2) model.

1) Let y_t be an ARMA(1,1) given by

$$y_t = \mu + \gamma y_{t-1} + \eta_t - \beta \eta_{t-1}$$

where η_t is a weak white noise. Then, we have

$$R^2(y_{t+1:t+n}, ARMA) = \frac{(1-\gamma^n)^2}{(1-\gamma)^2} \frac{Var[y_t]}{Var[y_{t+1:t+n}]} R^2(y_{t+1}, ARMA). \tag{A.3}$$

2) Let y_t be an ARMA(2,2) given by

$$y_t = \mu + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \eta_t + \theta_1 \eta_{t-1} + \theta_2 \eta_{t-2}$$

where η_t is a weak white noise. Then, we have

$$R^2(y_{t+1:t+n}, ARMA) = 1 - \frac{Var[y_{t+1:t+n} - BP[y_{t+1:t+n} \mid H_t(y)]]}{Var[y_{t+1:t+n}]}, \tag{A.4}$$

with

$$Var[y_{t+1:t+n} - BP[y_{t+1:t+n} \mid H_t(y)]] = \sum_{i=0}^{n-1} (\sum_{s=0}^{i} \psi_s)^2 Var[\eta_t], \tag{A.5}$$

where $\psi_i = A_1^\top \Phi^i A_2$, with *$A_1 = (1,0,0,0)^\top$, $A_2 = (1,0,1,0)^\top$,*

$$\Phi = \begin{bmatrix} \phi_1 & \phi_2 & \theta_1 & \theta_2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

and for a given second-order stationary process $\{z_t\}$, $H_t(z)$ denotes the Hilbert-space generated by $\{1, z_\tau, \tau \leq t\}$, while for a second-order stationary variable w, $BP[w \mid H_t(z)]$ denotes the best linear predictor of w given $H_t(z)$; i.e., $w = BP[w \mid H_t(z)] + \varepsilon$, with *$Cov(\varepsilon, x) = 0, \ \forall x \in H_t(z)$.*

Proof of Lemma A.2:

1) Define the process m_{t-1} by

$$m_{t-1} = \mu + \gamma y_{t-1} - \beta \eta_{t-1},$$

and let θ be the real $\theta = \mu(1-\gamma)^{-1}$. Hence, it follows that

$$m_{t-1} = \theta + \gamma(m_{t-2} - \theta) + (\gamma - \beta)\eta_{t-1}.$$

For $n \geq 2$, we have

$$BP[y_{t+n} \mid H_t(y)] = BP[m_{t+n-1} \mid H_t(y)] = \theta + \gamma BP[(m_{t+n-2} - \theta) \mid H_t(y)]$$

i.e.,

$$BP[y_{t+n} \mid H_t(y)] = \theta + \gamma^{n-1}(m_t - \theta),$$

which is also valid for $n = 1$. Thus,

$$BP[y_{t+1:t+n} \mid H_t(y)] = n\theta + \frac{1-\gamma^n}{1-\gamma}(m_t - \theta),$$

from which it follows that

$$\begin{aligned} R^2(y_{t+1:t+n}, ARMA) &= \frac{Var[BP[y_{t+1:t+n} \mid H_t(y)]]}{Var[y_{t+1:t+n}]} \\ &= \frac{(1-\gamma^n)^2}{(1-\gamma)^2} \frac{Var[m_t]}{Var[y_{t+1:t+n}]} \\ &= \frac{(1-\gamma^n)^2}{(1-\gamma)^2} \frac{Var[y_t]}{Var[y_{t+1:t+n}]} \frac{Var[m_t]}{Var[y_t]}, \end{aligned}$$

i.e., (A.3).

2) Following Baillie and Bollerslev (1992),

$$y_{t+n} - BP[y_{t+n} \mid H_t(y)] = \sum_{i=0}^{n-1} \psi_i \eta_{t+n-i},$$

for any $n > 0$. Therefore,

$$y_{t+1:t+n} - BP[y_{t+1:t+n} \mid H_t(y)] = \sum_{i=0}^{n-1} (\sum_{s=0}^{i} \psi_s) \eta_{t+n-i},$$

so that

$$Var[y_{t+1:t+n} - BP[y_{t+1:t+n} \mid H_t(y)]] = \sum_{i=0}^{n-1} (\sum_{s=0}^{i} \psi_s)^2 Var[\eta_t],$$

i.e., (A.5). Note that while the results of Baillie and Bollerslev (1992) are derived under the assumption that η_t is a martingale difference sequence and refers to $E[y_{t+n} \mid y_\tau, \tau \leq t]$ rather than $BP[y_{t+n} \mid H_t(y)]$, the final result remains valid when η_t is a weak white noise process (as for the integrated and realized volatilities); see also Meddahi and Renault (2002) and Meddahi (2002b) for further discussion along these lines.■

Proof of Proposition 4.1. Given that the unconditional mean of any eigenfunction $P_i(.)$, with $i \geq 1$, is zero, one gets (4.1). Let s be a positive real; then we have

$$\begin{aligned} E[\sigma^2_{t+s} \mid p_\tau, f_\tau, \tau \leq t] &= a_0 + \sum_{i=1}^{p} a_i E[P_i(f_{t+s}) \mid p_\tau, f_\tau, \tau \leq t] \\ &= a_0 + \sum_{i=1}^{p} a_i \exp(-\lambda_i s) P_i(f_t), \end{aligned} \tag{A.6}$$

which corresponds to (4.2) for $s = n$. By using (A.6),

$$\begin{aligned} E[IV_{t+n} \mid p_\tau, f_\tau, \tau \leq t] &= a_0 + \sum_{i=1}^{p} a_i \int_{t+n-1}^{t+n} E[P_i(f_u) \mid p_\tau, f_\tau, \tau \leq t] du \\ &= a_0 + \sum_{i=1}^{p} a_i \int_{t+n-1}^{t+n} \exp(-\lambda_i(u-t)) du \quad P_i(f_t) \\ &= a_0 + \sum_{i=1}^{p} a_i \exp(-\lambda_i(n-1)) \frac{(1-\exp(-\lambda_i))}{\lambda_i} \; P_i(f_t), \end{aligned}$$

i.e., (4.3). The result in (4.4) follows easily from the orthonormality of the eigenfunction, indicated in (3.5). In addition, for any real $s \geq 0$,

$$\begin{aligned} Var\left[\int_{t-1}^{t-1+s} \sigma^2_u du\right] &= E\left[\left(\sum_{i=1}^{p} a_i \int_{t-1}^{t-1+s} P_i(f_u) du\right)^2\right] \\ &= \sum_{1 \leq i,j \leq p} a_i a_i E\left[\int_{t-1}^{t-1+s} P_i(f_u) du \int_{t-1}^{t-1+s} P_j(f_u) du\right] \\ &= \sum_{1 \leq i,j \leq p} a_i a_i \delta_{ij} \frac{2}{\lambda_i^2}[\exp(-\lambda_i s) + \lambda_i s - 1], \end{aligned}$$

where the last equality follows from (A.1). Thus,

$$Var\left[\int_{t-1}^{t-1+s} \sigma^2_u du\right] = 2 \sum_{i=1}^{p} \frac{a_i^2}{\lambda_i^2}[\exp(-\lambda_i s) + \lambda_i s - 1], \tag{A.7}$$

which corresponds to (4.5) for $s = 1$. The result in (4.6) follows from the orthonormality of the eigenfunctions, equations (4.3), and the following equality:

$$Cov(IV_{t+n}, \sigma_t^2) = Cov(E[IV_{t+n} \mid p_\tau, f_\tau, \tau \le t], \sigma_t^2).$$

Equation (4.8) is obtained by a similar argument. Finally, for (4.7),

$$\begin{aligned}
Cov(IV_t, IV_{t+n}) &= Cov(IV_1, IV_{1+n}) \\
&= E\left[\left(\sum_{i=0}^{p} a_i \int_0^1 P_i(f_u)du\right)\left(\sum_{i=1}^{p} a_i \int_n^{n+1} P_i(f_u)du\right)\right] \\
&= \sum_{1\le i,j\le p} a_i a_j E\left[\int_0^1 P_i(f_u)du \int_1^{n+1} P_j(f_u)du\right] \\
&= \sum_{1\le i,j\le p} a_i a_j \delta_{ij} \exp(-\lambda_i(n-1)) \frac{[1-\exp(-\lambda_i)]^2}{\lambda_i^2} \\
&= \sum_{i=1}^{p} a_i^2 \exp(-\lambda_i(n-1)) \frac{[1-\exp(-\lambda_i)]^2}{\lambda_i^2},
\end{aligned}$$

where the second to last equality is obtained by using (A.2).■

Proof of Proposition 4.2. By straightforward calculations, it is easy to show that for any $\lambda > 0$,

$$[1-\exp(-\lambda)]^2 \ge \lambda^2 \exp(-\lambda), \tag{A.8}$$

$$\lambda \ge [1-\exp(-\lambda)], \tag{A.9}$$

$$2[\exp(-\lambda)+\lambda-1] \ge \lambda[1-\exp(-\lambda)], \tag{A.10}$$

$$\lambda^2 \ge 2[\exp(-\lambda)+\lambda-1]. \tag{A.11}$$

Thus, by using (A.8), it follows that for any $n \ge 1$,

$$\exp(-\lambda n) \le \exp(-\lambda(n-1)) \frac{[1-\exp(-\lambda)]^2}{\lambda^2},$$

so that

$$Cov(\sigma_{t+n}^2, \sigma_t^2) \le Cov(IV_{t+n}, IV_t).$$

Also, by using (A.9),

$$\frac{[1-\exp(-\lambda)]^2}{\lambda^2} \le \frac{[1-\exp(-\lambda)]}{\lambda} \le 1.$$

It follows therefore that for any $n \ge 1$,

$$\exp(-\lambda(n-1)) \frac{[1-\exp(-\lambda)]^2}{\lambda^2} \le \exp(-\lambda(n-1)) \frac{[1-\exp(-\lambda)]}{\lambda} \le 1,$$

and, hence,

$$Cov(IV_{t+n}, IV_t) \leq Cov(IV_{t+n}, \sigma_t^2).$$

The inequality (A.10) implies that for any $n \geq 1$,

$$\exp(-\lambda(n-1))\frac{[1-\exp(-\lambda)]}{\lambda} \leq \frac{[1-\exp(-\lambda)]}{\lambda} \leq 2\frac{[\exp(-\lambda)+\lambda-1]}{\lambda^2},$$

and, hence,

$$Cov(IV_{t+n}, \sigma_t^2) \leq Var[IV_t].$$

Finally, by (A.11),

$$Var[IV_t] \leq Var[\sigma_t^2],$$

which completes the proof of Proposition 4.2.■

Proof of (4.10). By definition, we have

$$R^2(IV_{t+1}, Best) = \frac{Var[E[IV_{t+1} \mid p_\tau, f_\tau, \tau \leq t]]}{Var[IV_{t+1}]}.$$

By using (4.3) for $n = 1$ along with the orthonormality of the eigenfunctions, i.e., (3.5), we get

$$Var[E[IV_{t+1} \mid p_\tau, f_\tau, \tau \leq t]] = \sum_{i=1}^{p} a_i^2 \frac{[1-\exp(-\lambda_i)]^2}{\lambda_i^2},$$

which combines to show (4.10).■

Proof of (4.11) and (4.12). By definition, we have

$$R^2(IV_{t+1}, \sigma_t^2) = \frac{Cov(IV_{t+1}, \sigma_t^2)^2}{Var[IV_{t+1}]Var[\sigma_t^2]}.$$

Thus, by using (4.6) for $n = 1$, we get (4.11). A similar argument results in 4.12).■

Proof of Proposition 4.3. Defining

$$q_i = \frac{a_i^2}{\sum_{i=1}^{p} a_i^2}, \eta_i = \frac{1-\exp(-\lambda_i)}{\lambda_i}, \text{ and } \phi_i = \frac{2(\exp(-\lambda_i)+\lambda_i-1)}{\lambda_i^2},$$

the following inequalities readily obtain,

$$0 < \eta_i^2 \leq \eta_i \leq \phi_i \leq 1.$$

From equations (4.10), (4.12), and Proposition 4.1, we have that

$$R^2(IV_{t+1}, IV_t) \leq R^2(IV_{t+1}, Best)$$

if and only if

$$\sum_{i=1}^{p} a_i^2 \frac{[1-\exp(-\lambda_i)]^2}{\lambda_i^2} \le \sum_{i=1}^{p} a_i^2 \frac{[1-\exp(-\lambda_i)]^2}{\lambda_i^2} Var[IV_t].$$

The above inequality may be written as,

$$\left(\sum_{i=1}^{p} q_i \eta_i^2\right)^2 \le \left(\sum_{i=1}^{p} q_i \eta_i^2\right)\left(\sum_{i=1}^{p} q_i \phi_i\right),$$

or equivalently,

$$\sum_{i=1}^{p} q_i \eta_i^2 \le \sum_{i=1}^{p} q_i \phi_i,$$

where the inequality holds element-by-element in the sum.

Likewise, from equation (4.11) and the above definitions, the inequality

$$R^2(IV_{t+1}, \sigma_t^2) \le R^2(IV_{t+1}, Best)$$

is tantamount to

$$\left(\sum_{i=1}^{p} q_i \eta_i\right)^2 \le \sum_{i=1}^{p} q_i \eta_i^2.$$

Letting the q_i define a discrete probability distribution and denoting the associated expectation operator $E_q[x] = \sum_{i=1}^{p} q_i x_i$ for $x = (x_1, .., x_p)$, this may be alternatively expressed as

$$E_q[\eta] \le (E_q[\eta^2])^{1/2},$$

which is simply Jensen's inequality. Note, that for $p = 1$, and thus $q = q_1 = 1$ and $\eta_1 = \eta \le 1$, this relation turns into the identity: $\eta = (\eta^2)^{1/2}$.

By similar arguments the relation

$$R^2(IV_{t+1}, IV_t) \le R^2(IV_{t+1}, \sigma_t^2)$$

holds if and only if

$$\left(\sum_{i=1}^{p} q_i \eta_i^2\right)^2 \le \left(\sum_{i=1}^{p} q_i \eta_i\right)^2 \left(\sum_{i=1}^{p} q_i \phi_i\right). \tag{A.12}$$

For $p = 1$, and thus again $q = q_1 = 1$ and $\eta_1 = \eta \leq 1$, this relation becomes

$$(\eta^2)^2 \leq \eta^2 \phi,$$

which again is valid from the noted inequalities above.

To illustrate the indeterminacy for $p = 2$, suppose that $q_1 = 0.9$ and $q_2 = 0.1$, while $\lambda_1 = 0.02$ and $\lambda_2 = 4$. The inequality in (A.12) then takes the form,

$$.7735 \leq .7711,$$

and hence it is violated. Changing only one parameter, $\lambda_2 = 1$, the relation is instead

$$.8504 \leq .8811,$$

and the inequality holds. Similar examples may be constructed for cases with a larger number of eigenfunctions, so there is no universal ranking between this pair of forecast procedures, except for the single eigenfunction case.■

Proof of (4.16) and (4.17). Both are simple applications of footnote 10.■

Proof of (4.18) and ARMA representations of integrated volatility. By definition, we have

$$R^2(IV_{t+1}, ARMA) = \frac{Var[BP[IV_{t+1} \mid H_t(IV)]]}{Var[IV_t]} = 1 - \frac{Var[IV_{t+1} - BP[IV_{t+1} \mid H_t(IV)]]}{Var[IV_t]},$$

i.e., (4.18) given that $IV_{t+1} - BP[IV_{t+1} \mid H_t(IV)]$ is the innovation of IV_{t+1} in its ARMA representation. We now derive the variance, denoted C_1, of this innovation (supporting results are given in Meddahi, 2002b).

1) When the spot variance depends on one eigenfunction, with a corresponding eigenvalue λ, Meddahi (2002b) shows that IV_t is an ARMA(1,1) with the following representation:

$$IV_t = (1 - \exp(-\lambda))a_0 + \exp(-\lambda) IV_{t-1} + \eta_t - \beta \eta_{t-1}, \qquad \text{(A.13)}$$

where η_t is a weak white noise with variance denoted C_1,

$$Var[\eta_t] = C_1 = \frac{C_2}{1 + \beta^2},$$

where

$$C_2 = (1 + \exp(-2\lambda)) Var[IV_t] - 2\exp(-\lambda) Cov(IV_t, IV_{t-1}), \;\; \beta = \frac{-1 + \sqrt{1 - 4\rho^2}}{2\rho}$$

$$\rho = \frac{C_3}{C_2}, \;\; C_3 = -\exp(-\lambda) Var[IV_t] + Cov(IV_t, IV_{t-1})$$

and $Var[IV_t]$ and $Cov(IV_t, IV_{t-1})$ are given respectively in (4.5) and (4.7).

2) When the spot variance depends on two eigenfunctions (with corresponding eigenvalues λ and $\tilde{\lambda}$), Meddahi (2002b) shows that IV_t is an ARMA(2,2) with the following representation:

$$\begin{aligned} IV_t = (1-\exp(-\lambda))(1-\exp(-\tilde{\lambda}))a_0 + (\exp(-\lambda)+\exp(-\tilde{\lambda}))IV_{t-1} \\ - \exp(-\lambda-\tilde{\lambda})IV_{t-2} + \eta_t - \beta_1\eta_{t-1} - \beta_2\eta_{t-2}, \end{aligned} \tag{A.14}$$

where η_t is a weak white noise with variance C_1,

$$C_1 = \frac{C_2}{1+\beta_1^2+\beta_2^2}$$

where

$$\begin{aligned} C_2 &= (1+\exp[-2(\lambda+\tilde{\lambda})] + (\exp[-\lambda]+\exp[-\tilde{\lambda}])^2)Var[IV_t] \\ &\quad - 2(\exp[-\lambda]+\exp[-\tilde{\lambda}])(1+\exp[-(\lambda+\tilde{\lambda})])Cov[IV_t, IV_{t-1}] \\ &\quad + 2\exp[-(\lambda+\tilde{\lambda})]Cov[IV_t, IV_{t-2}], \\ \beta_1 &= \frac{\beta_2}{1-\beta_2}\frac{\rho_1}{\rho_2}, \beta_2 = \frac{2s+1-\sqrt{4s+1}}{2s}, \rho_1 = \frac{C_3}{C_2}, \rho_2 = \frac{C_4}{C_2}, \\ C_3 &= -(1+\exp[-(\lambda+\tilde{\lambda})])(\exp[-\lambda]+\exp[-\tilde{\lambda}])Var[IV_t] \\ &\quad + (1+(\exp[-\lambda]+\exp[-\tilde{\lambda}])^2+\exp[-(\lambda+\tilde{\lambda})])Cov[IV_t, IV_{t-1}] \\ &\quad - (\exp[-\lambda]+\exp[-\tilde{\lambda}])Cov[IV_t, IV_{t-2}], \\ C_4 &= \exp[-(\lambda+\tilde{\lambda})])Var[IV_t] - (\exp[-\lambda]+\exp[-\tilde{\lambda}])Cov[IV_t, IV_{t-1}] + Cov[IV_t, IV_{t-2}], \\ s &= 2^{-1}\rho_2^2\rho_1^{-2}\left[-2-\rho_2^{-1}+sign(\rho_2)\sqrt{(2+\rho_2{}^{-1})^2-4\rho_1^2\rho_2^{-2}}\right], \end{aligned}$$

and $sign(\rho_2) = 1$ or -1 depending upon whether $\rho_2 > 0$ or $\rho_2 < 0$.■

Proof of Proposition 4.4. The result in (4.20) is a simple application of (4.1). By applying (4.3) we get:

$$\begin{aligned} E[IV_{t+1:t+n} \mid p_\tau, f_\tau, \tau \le t] &= na_0 + \sum_{s=1}^{n}\sum_{i=1}^{p} a_i \exp(-\lambda_i(s-1))\frac{[1-\exp(-\lambda_i)]}{\lambda_i}\ P_i(f_t) \\ &= na_0 + \sum_{i=1}^{p} a_i \left(\sum_{s=1}^{n}\exp(-\lambda_i(s-1))\right)\frac{[1-\exp(-\lambda_i)]}{\lambda_i}\ P_i(f_t) \\ &= na_0 + \sum_{i=1}^{p} a_i \frac{[1-\exp(-\lambda_i n)]}{\lambda_i}\ P_i(f_t), \end{aligned}$$

i.e., (4.21). Similarly, by using (4.6) one gets:

$$\begin{aligned} Cov(IV_{t+1:t+n}, \sigma^2_{t-l}) &= \sum_{s=1}^{n}\sum_{i=1}^{p} a_i^2 \exp(-\lambda_i(s+l-1))\frac{[1-\exp(-\lambda_i)]}{\lambda_i} \\ &= \sum_{i=1}^{p} a_i^2 \exp(-\lambda_i l)\left(\sum_{s=1}^{n}\exp(-\lambda_i(s-1))\right)\frac{[1-\exp(-\lambda_i)]}{\lambda_i} \\ &= \sum_{i=1}^{p} a_i^2 \frac{[1-\exp(-\lambda_i n)]}{\lambda_i}\ \exp(-\lambda_i l), \end{aligned}$$

i.e., (4.22). A similar proof establishes (4.23). Finally, (4.24) follows as a special case of (A.7) for $s = n$.■

Proof of (4.25), (4.26), (4.27), (4.28) and (4.29). By using Proposition 4.4, the proofs of (4.25), (4.26), (4.27), (4.28) and (4.29) are similar to the proofs of (4.10), (4.11), (4.12), (4.16) and (4.17) respectively.■

R^2s of Multi-step forecasts of Integrated Volatility. This is a simple application of Lemma A.2. The ARMA representation of integrated volatility is given in (A.13) when the spot variance depends on one eigenfunction and in (A.14) when the spot variance depends on two eigenfunctions.■

ARMA representations of realized volatility. Here, we give the ARMA representation of realized volatility provided in Meddahi (2002b). Note that the ARMA representation of the realized volatility $RV_t(h)$ is very similar to that of the integrated volatility. In particular, it has the same constant and autoregressive roots. There is a difference, however, in the variance of the innovation and the moving average roots.

1) When the spot variance depends on one eigenfunction, with a corresponding eigenvalue λ, Meddahi (2002b) shows that $RV_t(h)$ is an ARMA(1,1) with the following representation:

$$RV_t(h) = (1-\exp(-\lambda))a_0 + \exp(-\lambda)RV_{t-1}(h) + \eta_t(h) - \beta(h)\eta_{t-1}(h), \qquad (A.15)$$

where $\eta_t(h)$ is a weak white noise with variance denoted $C_1(h)$,

$$Var[\eta_t(h)] = C_1(h) = \frac{C_2(h)}{1+\beta^2(h)},$$

where

$$\begin{aligned}
C_2(h) &= (1+\exp(-2\lambda))Var[RV_t(h)] - 2\exp(-\lambda)Cov(RV_t(h), RV_{t-1}(h)), \\
\beta(h) &= \frac{-1+\sqrt{1-4\rho^2(h)}}{2\rho(h)}, \\
\rho(h) &= \frac{C_3(h)}{C_2(h)},\ C_3(h) = -\exp(-\lambda)Var[RV_t(h)] + Cov(RV_t(h), RV_{t-1}(h))
\end{aligned}$$

and $Var[RV_t(h)]$ and $Cov(RV_t(h), RV_{t-1}(h))$ are given respectively in (5.1) and (5.3).

2) When the spot variance depends on two eigenfunctions (with corresponding eigenvalues λ and $\tilde{\lambda}$), Meddahi (2002b) shows that $RV_t(h)$ is an ARMA(2,2) with the following representation:

$$\begin{aligned}
RV_t(h) = {}& (1-\exp(-\lambda))(1-\exp(-\tilde{\lambda}))a_0 + (\exp(-\lambda)+\exp(-\tilde{\lambda}))RV_{t-1}(h) \\
& - \exp(-\lambda-\tilde{\lambda})RV_{t-2}(h) + \eta_t(h) - \beta_1(h)\eta_{t-1}(h) - \beta_2(h)\eta_{t-2}(h),
\end{aligned} \tag{A.16}$$

where $\eta_t(h)$ is a weak white noise with variance $C_1(h)$,

$$C_1(h) = \frac{C_2(h)}{1+\beta_1^2(h)+\beta_2^2(h)}$$

where

$$\begin{aligned}
C_2(h) = {}& (1+\exp[-2(\lambda+\tilde{\lambda})] + (\exp[-\lambda]+\exp[-\tilde{\lambda}])^2)Var[RV_t(h)] \\
& - 2(\exp[-\lambda]+\exp[-\tilde{\lambda}])(1+\exp[-(\lambda+\tilde{\lambda})])Cov[RV_t(h), RV_{t-1}(h)] \\
& + 2\exp[-(\lambda+\tilde{\lambda})]Cov[RV_t(h), RV_{t-2}(h)],
\end{aligned}$$

$$\beta_1(h) = \frac{\beta_2(h)}{1-\beta_2(h)}\frac{\rho_1(h)}{\rho_2(h)}, \beta_2(h) = \frac{2s(h)+1-\sqrt{4s(h)+1}}{2s(h)}, \rho_1(h) = \frac{C_3(h)}{C_2(h)}, \rho_2(h) = \frac{C_4(h)}{C_2(h)},$$

$$\begin{aligned}
C_3(h) = {}& -(1+\exp[-(\lambda+\tilde{\lambda})])(\exp[-\lambda]+\exp[-\tilde{\lambda}])Var[RV_t(h)] \\
& + (1+(\exp[-\lambda]+\exp[-\tilde{\lambda}])^2 + \exp[-(\lambda+\tilde{\lambda})])Cov[RV_t(h), RV_{t-1}(h)] \\
& - (\exp[-\lambda]+\exp[-\tilde{\lambda}])Cov[RV_t(h), RV_{t-2}(h)], \\
C_4(h) = {}& \exp[-(\lambda+\tilde{\lambda})])Var[RV_t(h)] \\
& - (\exp[-\lambda]+\exp[-\tilde{\lambda}])Cov[RV_t(h), RV_{t-1}(h)] + Cov[RV_t(h), RV_{t-2}(h)], \\
s(h) = {}& 2^{-1}\rho_2^2(h)\rho_1^{-2}(h)\left[-2-\rho_2(h)^{-1} + sign(\rho_2(h))\sqrt{(2+\rho_2(h)^{-1})^2 - 4\rho_1^2(h)\rho_2^{-2}(h)}\right],
\end{aligned}$$

and $Var[RV_t(h)]$ and $Cov(RV_t(h), RV_{t-i}(h))$, $i = 1, 2$, are given respectively in (5.1) and (5.3).■

Proof of Proposition 5.1. Let y be a second-order stationary variable such that $Cov(y, U_t(h)) = 0$. Then:

$$R^2(y, RV_t(h)) = \frac{Cov(y, RV_t(h))^2}{Var[y]Var[RV_t(h)]} = \frac{Cov(y, IV_t + U_t(h))^2}{Var[y]Var[RV_t(h)]} = \frac{Cov(y, IV_t)^2}{Var[y]Var[IV_t]} \frac{Var[IV_t]}{Var[RV_t(h)]},$$

i.e., (5.5). Furthermore, for any $n \geq 1$ and $h > 0$, we have

$$BP[RV_{t+1:t+n}(h) \mid H_t(RV(h))] = BP[IV_{t+1:t+n} \mid H_t(RV(h))]$$

given that $RV_{t+i}(h) = IV_{t+i} + U_{t+i}(h)$, while $U_{t+i}(h)$, $i \geq 1$, is uncorrelated with any variable in $H_t(RV(h))$. Note that the same results holds if one considers the best predictor (of $RV_{t+1:t+n}(h)$ or $IV_{t+1:t+n}$) given lags of $RV_t(h)$ (and a constant). Hence,

$$BP[RV_{t+1:t+n}(h) \mid .] = BP[IV_{t+1:t+n} \mid .],$$

and as a consequence,

$$\begin{aligned} R^2(RV_{t+1:t+n}(h), .) &= \frac{Var[BP[RV_{t+1:t+n}(h) \mid .]]}{Var[RV_{t+1:t+n}(h)]} = \frac{Var[BP[IV_{t+1:t+n} \mid .]]}{Var[RV_{t+1:t+n}(h)]} \\ &= \frac{Var[BP[IV_{t+1:t+n} \mid .]]}{Var[IV_{t+1:t+n}]} \frac{Var[IV_{t+1:t+n}]}{Var[RV_{t+1:t+n}(h)]} \\ &= R^2(IV_{t+1:t+n}, .) \frac{Var[IV_{t+1:t+n}]}{Var[IV_{t+1:t+n}] + nVar[U_t(h)]}, \end{aligned}$$

i.e., (5.6).■

Table 1
Ideal one-period-ahead forecasts of integrated volatility

Model	M1	M2	M3
$R^2(IV_{t+1}, Best)$	.977	.830	.989
$R^2(IV_{t+1}, \sigma_t^2)$	.977	.819	.989
$R^2(IV_{t+1}, \sigma_t^2, 1)$	.977	.820	.989
$R^2(IV_{t+1}, \sigma_t^2, 4)$	.977	.821	.989
$R^2(IV_{t+1}, IV_t)$	.955	.689	.977
$R^2(IV_{t+1}, IV_t, 1)$	.957	.694	.979
$R^2(IV_{t+1}, IV_t, 4)$	.957	.698	.979
$R^2(IV_{t+1}, ARMA)$	.957	.699	–

Table 2
Ideal multi-period-ahead forecasts of integrated volatility

Model	M1			M2			M3		
Horizon	5	10	20	5	10	20	5	10	20
$R^2(IV_{t+1:t+n}, Best)$	.891	.797	.645	.586	.479	.338	.945	.895	.807
$R^2(IV_{t+1:t+n}, \sigma_t^2)$	.891	.797	.645	.492	.349	.222	.945	.894	.804
$R^2(IV_{t+1:t+n}, \sigma_t^2, 1)$	.891	.797	.645	.499	.359	.231	.945	.894	.804
$R^2(IV_{t+1:t+n}, \sigma_t^2, 4)$	.891	.797	.645	.508	.371	.242	.945	.894	.804
$R^2(IV_{t+1:t+n}, IV_t)$	.871	.779	.630	.445	.320	.214	.934	.885	.796
$R^2(IV_{t+1:t+n}, IV_t, 1)$	.873	.781	.632	.445	.330	.216	.936	.886	.796
$R^2(IV_{t+1:t+n}, IV_t, 4)$	.874	.781	.632	.446	.343	.227	.936	.886	.797
$R^2(IV_{t+1:t+n}, ARMA)$	.874	.781	.632	.460	.347	.231	–	–	–

Table 3

One-period-ahead forecasts of integrated volatility based on realized volatility

Model	M1			M2			M3		
1/h	48	96	288	48	96	288	48	96	288
$R^2(IV_{t+1}, RV_t(h))$	.836	.891	.932	.476	.563	.641	.881	.927	.960
$R^2(IV_{t+1}, RV_t(h), 1)$	.873	.906	.934	.507	.574	.642	.917	.943	.962
$R^2(IV_{t+1}, RV_t(h), 4)$	.883	.908	.934	.519	.580	.642	.929	.946	.963
$R^2(IV_{t+1}, RV_t(h), ARMA)$	.883	.908	.934	.522	.582	.646	-	-	-

Table 4

Multi-period-ahead forecasts of integrated volatility based on realized volatility

Horizon	5			10			20		
1/h	48	96	288	48	96	288	48	96	288
Model M1									
$R^2(IV_{t+1:t+n}, RV_t(h))$	.762	.813	.851	.682	.727	.761	.551	.588	.615
$R^2(IV_{t+1:t+n}, RV_t(h), 1)$	.797	.827	.852	.713	.740	.762	.576	.598	.616
$R^2(IV_{t+1:t+n}, RV_t(h), 4)$	.805	.829	.852	.720	.741	.762	.582	.599	.616
$R^2(IV_{t+1:t+n}, RV_t(h), ARMA)$	.806	.829	.852	.721	.741	.762	.582	.599	.616
Model M2									
$R^2(IV_{t+1:t+n}, RV_t(h))$	.307	.364	.414	.226	.268	.305	.148	.175	.199
$R^2(IV_{t+1:t+n}, RV_t(h), 1)$	.339	.381	.419	.255	.285	.312	.169	.188	.205
$R^2(IV_{t+1:t+n}, RV_t(h), 4)$	.360	.395	.429	.277	.302	.325	.186	.202	.216
$R^2(IV_{t+1:t+n}, RV_t(h), ARMA)$	.368	.400	.434	.286	.309	.330	.194	.208	.221
Model M3									
$R^2(IV_{t+1:t+n}, RV_t(h))$	.843	.886	.918	.797	.839	.869	.717	.754	.781
$R^2(IV_{t+1:t+n}, RV_t(h), 1)$	.877	.901	.920	.830	.853	.871	.747	.768	.783
$R^2(IV_{t+1:t+n}, RV_t(h), 4)$	.889	.904	.920	.841	.856	.871	.757	.770	.784

Table 5

One-period-ahead forecasts of realized volatility based on realized volatility

Model	M1			M2			M3		
1/h	48	96	288	48	96	288	48	96	288
$R^2(RV_{t+1}(h), RV_t(h))$	.731	.832	.911	.328	.460	.597	.795	.879	.943
$R^2(RV_{t+1}(h), RV_t(h), 1)$	.765	.846	.912	.350	.469	.597	.827	.894	.945
$R^2(RV_{t+1}(h), RV_t(h), 4)$	.773	.848	.912	.358	.474	.600	.838	.897	.945
$R^2(RV_{t+1}(h), RV_t(h), ARMA)$	.773	.848	.912	.360	.475	.601	-	-	-

Table 6

Multi-period-ahead forecasts of realized volatility based on realized volatility

Horizon	5			10			20		
1/h	48	96	288	48	96	288	48	96	288
Model M1									
$R^2(RV_{t+1:t+n}(h), RV_t(h))$	.740	.801	.847	.671	.722	.759	.546	.585	.614
$R^2(RV_{t+1:t+n}(h), RV_t(h), 1)$	.774	.815	.848	.702	.734	.760	.571	.595	.615
$R^2(RV_{t+1:t+n}(h), RV_t(h), 4)$	.782	.816	.848	.709	.735	.760	.577	.597	.615
$R^2(RV_{t+1:t+n}(h), RV_t(h), ARMA)$	.782	.816	.848	.709	.735	.760	.577	.597	.615
Model M2									
$R^2(RV_{t+1:t+n}(h), RV_t(h))$	.274	.343	.406	.210	.258	.302	.140	.170	.197
$R^2(RV_{t+1:t+n}(h), RV_t(h), 1)$	.303	.359	.410	.237	.275	.308	.160	.184	.203
$R^2(RV_{t+1:t+n}(h), RV_t(h), 4)$	.321	.372	.421	.258	.291	.321	.177	.197	.214
$R^2(RV_{t+1:t+n}(h), RV_t(h), ARMA)$	.328	.378	.425	.266	.297	.326	.184	.202	.219
Model M3									
$R^2(RV_{t+1:t+n}(h), RV_t(h))$	.824	.876	.914	.788	.834	.867	.713	.752	.781
$R^2(RV_{t+1:t+n}(h), RV_t(h), 1)$	.858	.891	.917	.821	.848	.869	.742	.765	.783
$R^2(RV_{t+1:t+n}(h), RV_t(h), 4)$	.869	.895	.917	.832	.851	.869	.752	.768	.783

Table 7

Forecasts of integrated and realized volatilities from past daily squared returns

	Model	M1				M2				M3			
	Horizon	1	5	10	20	1	5	10	20	1	5	10	20
IV	lag												
	0	.122	.111	.100	.081	.031	.020	.015	.010	.157	.150	.142	.128
	1	.210	.191	.171	.138	.048	.033	.025	.017	.266	.255	.241	.217
	4	.360	.329	.294	.238	.072	.054	.043	.029	.452	.432	.409	.369
	19	.493	.450	.402	.325	.092	.074	.061	.043	.639	.611	.580	.523
	39	.498	.454	.406	.328	.093	.075	.062	.044	.653	.625	.593	.535
	GARCH	.498	.454	.406	.328	.093	.075	.062	.044	-	-	-	-
1/h	lag												288
288	0	.119	.110	.099	.080	.029	.019	.015	.009	.154	.149	.142	.128
	1	.205	.190	.171	.138	.044	.032	.024	.016	.261	.254	.241	.217
	4	.352	.327	.293	.238	.066	.052	.042	.029	.444	.430	.408	.369
	19	.482	.448	.401	.325	.085	.072	.060	.043	.628	.609	.579	.522
	39	.486	.452	.405	.328	.086	.074	.061	.043	.641	.623	.592	.534
	GARCH	.486	.452	.405	.328	.086	.074	.061	.043	-	-	-	-
96	0	.114	.109	.099	.080	.025	.019	.014	.009	.149	.148	.141	.128
	1	.196	.188	.170	.137	.039	.031	.024	.016	.252	.252	.240	.216
	4	.336	.324	.292	.237	.058	.050	.040	.028	.429	.427	.407	.368
	19	.460	.443	.399	.324	.075	.069	.058	.042	.606	.604	.577	.521
	39	.465	.447	.403	.327	.076	.070	.059	.043	.619	.618	.590	.533
	GARCH	.465	.447	.403	.327	.076	.070	.059	.043	-	-	-	-
48	0	.107	.108	.098	.080	.021	.018	.014	.009	.142	.147	.140	.127
	1	.184	.185	.168	.137	.033	.029	.023	.016	.240	.249	.238	.216
	4	.315	.319	.289	.236	.049	.048	.039	.029	.408	.423	.404	.367
	19	.432	.437	.396	.322	.063	.066	.056	.042	.576	.598	.573	.520
	39	.436	.441	.400	.325	.064	.067	.057	.043	.589	.611	.586	.532
	GARCH	.436	.441	.400	.325	.064	.067	.057	.043	-	-	-	-
1	0	.016	.046	.057	.057	.001	.003	.003	.003	.026	.073	.092	.099
	1	.027	.079	.098	.098	.002	.005	.005	.005	.043	.123	.156	.168
	4	.046	.136	.168	.167	.003	.008	.009	.008	.073	.209	.264	.286
	19	.063	.185	.229	.229	.004	.011	.013	.012	.103	.295	.374	.405
	39	.064	.187	.232	.231	.004	.012	.014	.013	.105	.302	.383	.415
	GARCH	.064	.187	.232	.231	.004	.012	.014	.013	-	-	-	-

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