Moments of Continuous Time Stochastic Volatility Models*

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Abstract

In this paper, we derive explicit formulas for conditional and unconditional moments of the continuous time Eigenfunction Stochastic Volatility (ESV) models of Meddahi (2001). Special cases of ESV models are log-normal, affine and GARCH diffusion models. The conditional moment restrictions we derive are based only on observable variables. Therefore, using an instrumental variable approach is easy. A major advantage of this approach with respect to other methods (e.g., simulation techniques) is that one can use extra variables as instruments, in particular Black-Scholes implied volatilities (without specifying a price of risk) and high-frequency realized volatility. Moment conditions are also useful for statistical inference purposes. This is of interest given that we observe many financial derivative prices. Therefore, one can test which features of the model are or are not compatible with financial derivatives.

Key words: Continuous time, stochastic volatility, eigenfunction, moments, instruments, implied volatility, GMM.

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1 Introduction and motivations

In this paper, we derive explicit formulas of several moments of continuous time stochastic volatility models, including Log-normal models of Melino and Turnbull (1990), the square-root model of Heston (1993), the affine models of Duffie, Pan and Singleton (2000) and the GARCH diffusion model of Nelson (1990). These models are special cases of the general framework considered in a companion paper, Meddahi (2001), and termed Eigenfunctions Stochastic Volatility (ESV, hereafter) models.[1] This class is characterized by the fact that the variance process is assumed to be a linear combination of the eigenfunctions of the infinitesimal generator of the state variable governing the model. We derive unconditional moments, centered and noncentered, the covariance structures of the returns, the squared returns, and the covariance between the returns and the squared returns, which is a measure of skewness and leverage effect. Such moments are useful for inference purposes by the Generalized Method of Moment (GMM) of Hansen (1982).

The most popular inference methods in the stochastic volatility literature are those based on simulation techniques.[2] The reason is that the variance depends on an unobservable state variable and hence non-simulated methods need (numerical) integration, which is difficult in the discrete time case (e.g., Danielsson and Richard, 1994) and unfeasible in the continuous time case.[3] Simulated methods estimation of continuous time SV models include the simulated method of moments (SMM) of Duffie and Singleton (1993), the Indirect Inference (II) of Gouriéroux, Monfort and Renault (1993) and the Efficient Method of Moments (EMM) of Gallant and

[1]For a review of volatility models, see Bollerslev, Engle and Nelson (1994), Ghysels, Harvey and Renault (1996) and Shephard (1996).

[2]There are many estimation methods in the literature for continuous time models: the GMM of Hansen and Scheinkman (1995) for equally spaced data and the extension to unequally spaced data of Duffie and Glynn (1997); the regression-type nonparametric methods of Ait-Sahalia (1996), Stanton (1997) and Bandi and Phillips (1999); the eigenfunction-type nonparametric methods of Darroles, Florens and Gouriéroux (1998) and Chen, Hansen and Scheinkman (1998); the maximum likelihood approximation approach of Ait-Sahalia (2002); the Bayesian methods of Elerian, Chib and Shephard (2001), Eraker (2001) and Jones (1998). In the specific affine models of, e.g., Duffie, Pan and Singleton (2000), estimation techniques based on the characteristic function are developed by Chacko and Viceira (1999), Singleton (2001) and Carrasco et al. (2001).

[3]In the affine case, Chako and Viceira (1999) integrate over the state variable by neglecting the dependence between the variance and the observables wich is, in general, a strong assumption. The Bayesian methods may potentially be extended to the SV case. Indeed, Eraker (1998) does this. However, the variance is assumed to be log-normal and extension to general SV case is unclear. Finally, Hansen and Scheikman (1995) method can be applied to subordinated observation which is a specific SV-type modeling; see the application to interest rates by Conley et al. (1997).

Tauchen (1996); For a review of simulated methods in financial econometrics, see Gouriéroux and Monfort (1996), and Tauchen (1997).

A potential limitation of the GMM based on ad hoc unconditional moments is their finite sample properties. In particular, Jacquier, Polson and Rossi (1994) and Andersen and Sorensen (1996) show (by Monte Carlo experiments) that for discrete time log-normal SV models, the finite sample properties of the GMM estimators are poor especially when the number of moments is high (Tauchen, 1986). Moreover, Andersen, Chung and Sorensen (1997) show that the EMM estimators dominate the GMM ones in the discrete time log-normal SV model, while Gallant and Tauchen (1999) show it for some continuous time models. However, there are many reasons why deriving GMM estimates are useful. We give six reasons.

i) The aforementioned simulated methods need a starting value of the unknown parameter in the simulation experiment. Since these methods are time-consuming, especially in the multivariate case, a GMM estimator may be used.

ii) The main problem of the GMM estimators is that they have a finite sample bias. However several reduced bias techniques are available: the continuous updating GMM of Hansen, Heaton and Yaron (1996) and the recent literature on empirical likelihood methods (e.g., Kitamura, 1997); see Newey and Smith (2000).

iii) The performance of II and EMM (as well as the other methods) in the multivariate case is not very clear. To the best of our knowledge, there is no application of II in the multivariate case while EMM is done only with two variables (e.g., Gallant, Hsu and Tauchen, 1998) or three variables (Dai and Singleton, 2000). However, the GMM method adds no difficulty in the multivariate case. Indeed, recently, Bakschi, Cao and Chen (1997), Benzoni (2000) and Chernov (2000) among others, consider multivariate models and use the SMM of Duffie and Singleton (1993) for the inference.

iv) II and EMM provide diagnostics based on the estimation of the auxiliary model. However, additional diagnostic tests may be useful, to see, for instance, what are the characteristics of the data (not considered in the auxiliary model) that the structural model captures well or no. It turns out that GMM provides easy diagnostics by using the M-tests of Newey (1985) and Tauchen (1985). Observe also that with explicit formulae of the moments, one can also use diagnostics as in Das and Sundaram (1999).

v) Using the GMM method is very useful for robustness against misspecification. As advocated by Conley, Hansen and Liu (1997), it is important to do inference based on the steady-state distribution (i.e., the long-run), which is typically related to some unconditional moments, for instance the first marginal moments. The reason is that all the considered models are approximation of the reality. As a consequence, developing robust inference methods for estimating some parameter of interest may be useful. It turns out that the ESV models of Meddahi (2001) have some interesting robust properties. More precisely, some marginal moments as well as some correlations of the returns and the squared returns are the same for many models. For instance, while the state variable that governs the variance process may be an Ornstein-Uhlenbeck process or a square-root one, some moments are the same. As a consequence, one can estimate by GMM the parameters identified by these moments robustly against potential misspecification. However, II of Gouriéroux, Monfort and Renault (1993) and EMM of Gallant and Tauchen (1996) use all the feature of the model. Therefore, a priori, they are not robust to misspecification (see however, Dhaene, Gourieroux and Scaillet, 1998, Dridi and Renault, 2000, and Gallant, 2002).

vi) We also derive conditional moment restrictions. These conditional restrictions are based on observable variables only. More precisely, we derive multi-period conditional moment restrictions (Hansen, 1985). As a consequence, one can use the optimal instruments developed in that case (Hansen 1985; Hansen, Heaton and Ogaki, 1988; Hansen and Singleton, 1996). More importantly, one can use extra variables as instruments, in particular Black-Scholes implied volatilities, without specifying a price of risk. This is of interest given that we observe many financial derivative prices. Therefore, one can test which features of the model are or are not compatible with financial derivatives. Potential other instruments are high-frequency realized volatilities (see Andersen, Bollerslev and Diebold, 2001, and Barndorff-Nielsen and Shephard, 2002), range observations, volume of transactions, etc. This reason is the major advantage of the GMM approach with respect to other methods (e.g., simulation techniques).

Computing moments of continuous time SV models is already considered in the literature in some cases. To the best of our knowledge, it is done only in the affine case, by Das and Sundaram (1999), and Pan (2001). Das and Sundaram (1999) compute the four first moments for the Heston model without leverage effect. Pan (2001) computes conditional moment restrictions based on returns and unobservable volatility in the square-root model with leverage effect and jumps. Pan (2001) uses the options data in her inference since the information contained in the stock returns and the variance process is the same that the one contained in the stock returns and the options data. Both Das and Sundaram (1999) and Pan (2001) compute the generating function of the bivariate vector of the returns and the variance processes and then derive the moments. A crucial assumption in their framework is that the model is affine, i.e. all the drifts, variances, covariances and jump-intensity are an affine function of the variance process assumed to be a square-root process. An additional assumption in Pan (2001) is that all the risk premia functions are affine functions of the variance. However, in our framework, we do not need that the variance process is a square-root model. Moreover, in this case, the drifts, variances, covariances, and risk premia may be non-linear functions of the variances. We need square-integrable functions.

We compute the moments by expanding some functions onto the eigenfunctions of the infinitesimal generator of the state variable driving the volatility. Kessler and Sorensen (1999) used the same approach for scalar diffusion models. Thus, our results are a generalization of Kessler and Sorensen (1999) to the stochastic volatility case. In addition, Sorensen (2000) proposed an estimating function method to estimate scalar diffusion models by using Kessler and Sorensen (1999). Estimating function method is very close to the GMM.

The paper is organized as follows. In section 2, we recap the main properties of the ESV models of Meddahi (2001). In the third section, we compute the moments for the model without leverage effect while the fourth section deals with the leverage case. The last section concludes, while all the proofs are provided in the Appendix.

2 Eigenfunction Stochastic Volatility Models

In this section, we recap the main properties of the Eigenfunction Stochastic Volatility (ESV) models introduced in Meddahi (2001). These models provide a convenient tractable framework, where many well-known models can be represented and in which analytic calculations can readily be performed. We will give a brief introduction to the general class of models, before indicating how common volatility models can be rewritten in this form.

2.1 General theory

The most popular stochastic volatility models like log-normal (Hull and White, 1987; Wiggins, 1987), square-root (Heston, 1993) and GARCH diffusion (Nelson, 1990) models have the following form:

$$d\log(S_t) = m_t dt + \sigma_t \; [\sqrt{1-\rho^2} dW_t^{(1)} + \rho dW_t^{(2)}], \quad \text{with} \tag{2.1}$$

$$\sigma_t^2 = g(f_t).$$

Here f_t is a state variable with simple dynamics that is characterized by

$$df_t = \mu(f_t)dt + \sigma(f_t)dW_t^{(2)}; \tag{2.2}$$

$g(.)$ is a known and *ad hoc* function; and $W_t^{(1)}$ and $W_t^{(2)}$ are two independent standard Brownian processes. In particular, we can represent:
1- Log-normal model: $\sigma_t^2 = \exp(f_t), \quad df_t = k[\theta - f_t]dt + \sigma dW_t^{(2)}$;
2- Square-root model: $\sigma_t^2 = f_t, \quad df_t = k[\theta - f_t]dt + \sigma\sqrt{f_t} dW_t^{(2)}$;
3- GARCH diffusion model: $\sigma_t^2 = f_t, \quad df_t = k[\theta - f_t]dt + \sigma f_t dW_t^{(2)}$.

Instead of taking an *ad hoc* function $g(.)$, Meddahi (2001) proposes a flexible functional approach. More precisely, he assumes that the variance process σ_t^2 is given by

$$\sigma_t^2 = \sum_{i=0}^{p} a_i E_i(f_t), \tag{2.3}$$

where p is an integer, potentially infinite; a_i, $i = 0, ..., p$, are real numbers; and $E_i(f_t)$ are the eigenfunctions of the infinitesimal generator associated with f_t; see Hansen, Scheinkman (1995) and Ait-Sahalia, Hansen and Scheinkman (2001) for a review.

We now recap the definition of this operator and some related properties. Let $\mathcal{A}$ be the infinitesimal generator operator associated with f_t:

$$\mathcal{A}\phi(f_t) \equiv \mu(f_t)\phi'(f_t) + \frac{\sigma^2(f_t)}{2}\phi''(f_t), \tag{2.4}$$

where $\phi(f_t)$ is a square-integrable function and twice differentiable. Let $E_i(f_t)$, $i = 0, 1, ...$, be the set of the eigenfunctions of $\mathcal{A}$ with corresponding eigenvalues $(-\delta_i)$, i.e.

$$\mathcal{A}E_i(f_t) = -\delta_i E_i(f_t). \tag{2.5}$$

Here, we assume that the eigenvalues are real numbers and that the spectrum, i.e. the set of the eigenvalues, is discrete:

Assumption A1. The stationary process $\{f_t\}$ is time reversible.
Assumption A2. The spectrum of the infinitesimal generator operator $\mathcal{A}$ of $\{f_t\}$ is discrete and denoted $\{-\delta_i, i \in \mathbf{N}\}$ with $\delta_0 = 0$ and $\delta_0 < \delta_1 < \delta_2 < ... < \delta_i < \delta_{i+1}...$; the corresponding eigenfunctions are denoted $E_i(f_t)$, $i \in \mathbf{N}$.
Hansen, Scheinkman and Touzi (1998) show that under some appropriate boundary protocol, stationary scalar diffusions are time-reversible. Hence, assumption **A1** is not restrictive when one considers a volatility model that depends on one factor. It is, however, when one considers a multivariate vector f_t. This assumption is ensured when the factors are independent as in the volatility literature. Assumption **A2** is true for both log-normal and square-root models but not for the GARCH diffusion model. A sufficient assumption that ensures **A2** is that the operator $\mathcal{A}$ is compact.

The eigenfunctions have some interesting properties:
i) two eigenfunctions $E_i(f_t)$ and $E_j(f_t)$ associated with two different eigenvalues are orthogonal, and any nonconstant eigenfunction is centered:

$$E[E_i(f_t)E_j(f_t)] = 0 \text{ and } E[E_i(f_t)] = 0; \tag{2.6}$$

ii) any eigenfunction is an autoregressive process of order one, in general heteroskedastic:

$$\forall h > 0,\ E[E_i(f_{t+h}) \mid f_\tau, \tau \leq t] = \exp(-\delta_i h)E_i(f_t); \tag{2.7}$$

iii) any square-integrable function g, i.e. $E[g(f_t)^2] < \infty$, may be written as a linear combination of the eigenfunctions, i.e.

$$g(f_t) = \sum_{i=0}^{\infty} a_i E_i(f_t) \quad \text{where} \quad a_i = E[g(f_t)E_i(f_t)] \quad \text{and} \quad \sum_{i=0}^{\infty} a_i^2 = E[g(f_t)^2] < \infty. \tag{2.8}$$

Therefore, $g(f_t)$ is the limit in mean-square of $\sum_{i=0}^{p} a_i E_i(f_t)$ when p goes to $+\infty$.[4]

These three properties explain the powerfullness of the ESV approach. Consider any function of current or future values of returns. Given the Markovian nature of the joint process $(Log(S_t), f_t)$, a conditional expectation of any transformation of this variable, like the variance, is a function of f_t. Therefore, by using the third property, one can expand this function onto the eigenfunctions. The autoregressive features of these eigenfunctions (second property) allow for ready computation of the dynamics of this function. Finally, given the first property, it is easy to compute the covariance of two functions.

[4]Observe that we make a normalization assumption by specifying that $Var[E_i(f_t)] = 1$ for $i \neq 0$. Likewise, we assume that $E_0(f_t) = 1$.

2.2 Examples

2.2.1 The log-normal example

Consider the state variable f_t defined by, after a normalization,

$$df_t = -kf_t dt + \sqrt{2k}\; dW_t^{(2)}. \tag{2.9}$$

The eigenfunction associated with the Ornstein-Uhlenbeck process (2.9) are the Hermite polynomials H_i associated with the eigenvalues $\delta_i = ki$. These polynomials are characterized by

$$H_0(x) = 1,\; H_1(x) = x \text{ and } \forall i > 1, H_i(x) = \frac{1}{\sqrt{i}}\{xH_{i-1}(x) - \sqrt{i-1}H_{i-2}(x)\}. \tag{2.10}$$

Meddahi (2001) shows that the log-normal model of Hull and White (1987) and Wiggins (1987) is an ESV model with

$$\sigma_t^2 = \sum_{i=0}^{\infty} a_i H_i(f_t), \quad \text{where} \;\; a_i = \exp(\theta + \frac{\sigma^2}{4k})\frac{(\sigma/\sqrt{2k})^i}{\sqrt{i!}}. \tag{2.11}$$

2.2.2 The square-root example

Consider the state variable f_t defined by, after a normalization,

$$df_t = k(\alpha + 1 - f_t)dt + \sqrt{2k}\sqrt{f_t}dW_t^{(2)} \text{ with } \;\alpha = \frac{2k\theta}{\eta^2} - 1. \tag{2.12}$$

The eigenfunctions associated with (2.12) are the Laguerre polynomials $L_i^{(\alpha)}(f_t)$ associated with the eigenvalues $\delta_i = ki$. The Laguerre polynomials are characterized by:

$$\begin{aligned} \begin{pmatrix} i+\alpha \\ i \end{pmatrix}^{1/2} i\; L_i^{(\alpha)}(x) &= \begin{pmatrix} i-1+\alpha \\ i-1 \end{pmatrix}^{1/2} (-x + 2i + \alpha - 1)L_{i-1}^{(\alpha)}(x) \\ &- \begin{pmatrix} i-2+\alpha \\ i-2 \end{pmatrix}^{1/2} (i + \alpha - 1)L_{i-2}^{(\alpha)}(x), \text{ where} \end{aligned} \tag{2.13}$$

$$L_0^{(\alpha)}(x) = 1, \quad L_1^{(\alpha)}(x) = \frac{1 + \alpha - x}{\sqrt{1+\alpha}}.$$

Meddahi (2001) shows that the square-root model of Heston (1993) is an ESV model with

$$\sigma_t^2 = a_0 L_0^{(\alpha)}(f_t) + a_1 L_1^{(\alpha)}(f_t) \quad \text{where} \;\; a_0 = \theta \;\; \text{and} \;\; a_1 = -\frac{\sqrt{\theta}\eta}{\sqrt{2k}}. \tag{2.14}$$

Note that this is also the case for the affine model of Duffie, Pan and Singleton (2000).

2.2.3 The GARCH diffusion example

Consider the state variable f_t defined by

$$df_t = k(\theta - f_t)dt + \sigma f_t dW_t^{(2)}. \tag{2.15}$$

This process was first introduced by Wong (1964) and popularized by Nelson (1990). This process violates assumption **A2**. The main consequence is that in the expansion results (third property), one has to take an integral instead of a sum. We will not consider this approach in this paper. Instead, we assume that the variance is a GARCH diffusion model, i.e. $g(x) = x$, and that the second moment of the variance σ_t^2 is finite. These assumptions suffice to do all the calculations, since the first eigenfunction is an affine function given by

$$E_1(x) = \frac{\sqrt{1-\lambda}}{\theta\sqrt{\lambda}}(x - \theta) \quad \text{where} \quad \lambda = \sigma^2/2k, \tag{2.16}$$

and the variance depends only on E_0 and E_1. Indeed, we have:

$$\sigma_t^2 = a_0 E_0(f_t) + a_1 E_1(f_t) \quad \text{where} \quad a_0 = \theta \quad \text{and} \quad a_1 = \frac{\theta\sqrt{\lambda}}{\sqrt{1-\lambda}}. \tag{2.17}$$

Note that the second moment of the variance σ_t^2 is finite when λ is smaller than one. Andersen and Bollerslev (1998) and Andreou and Ghysels (2001) who consider this example also assume the existence of the second moment of σ_t^2 in order to use the weak GARCH results of Drost and Werker (1996).

2.3 The multifactor case

Meddahi (2001) considers also the case where the variance is a function of several factors as in Bollerslev and Zhou (2001), Engle and Lee (1999) and Harvey, Ruiz and Shephard (1994) among others. Without loss of generality, we consider the two-factor case. Let $f_{1,t}$ and $f_{2,t}$ be two independent stochastic processes characterized by

$$df_{j,t} = \mu_j(f_{j,t})dt + \sigma_j(f_{j,t})dW_{j,t}, \; j = 1, 2, \tag{2.18}$$

where the eigenfunctions (resp eigenvalues) of the corresponding infinitesimal generator are denoted $E_{1,i}(f_{1,t})$ and $E_{2,i}(f_{2,t})$ (resp $\delta_{1,i}$ and $\delta_{2,i}$). Then the variance process σ_t^2 is defined by

$$\sigma_t^2 = \sum_{0 \le i,j \le p} a_{i,j} E_{1,i}(f_{1,t}) E_{2,j}(f_{2,t}) \quad \text{where} \quad \sum_{0 \le i,j \le p} a_{i,j}^2 < \infty.$$

It turns out that the properties of the eigenfunctions defined in (2.6), (2.7) and (2.8) also hold for the functions $E_{i,j}(f_t)$ defined by

$$E_{i,j}(f_t) \equiv E_{1,i}(f_{1,t})E_{2,j}(f_{2,t}) \quad \text{where} \quad f_t \equiv (f_{1,t}, f_{2,t})'. \tag{2.19}$$

Hence, $E_{i,j}(f_t)$ are the eigenfunctions associated with the bivariate state variable $(f_{1,t}, f_{2,t})$.[5]

2.4 Some notations and expansions

In the following propositions, we expand some functions onto the eigenfunctions for computation purposes. For each case, we assume that the (L^2) expansion is feasible, i.e., the function of interest is square-integrable.
1) For a given i, the reals $\{e_{i,j}\}$ and p_i are defined by

$$\sigma_t \sigma(f_t) E_i'(f_t) = \sum_{j=0}^{p_i} e_{i,j} E_j(f_t), \tag{2.20}$$

where $E_i'(.)$ is the first derivative of $E_i(.)$.
2) For a given i and j, the reals $\{w_{i,j,k}\}$ and $p_{i,j}$ are defined by

$$E_i(f_t)E_j(f_t) = \sum_{k=0}^{p_{i,j}} w_{i,j,k} E_k(f_t). \tag{2.21}$$

3 The baseline model

In this section, we consider a process $\{y_t, t \in I\!R^+\}$ given by

$$dy_t = \sigma_t \, dW_t^{(1)}, \text{ with} \tag{3.1}$$

$$\sigma_t^2 = \sum_{i=0}^{p} a_i E_i(f_t), \qquad p \in \mathbf{N} \bigcup \{+\infty\}, \qquad \sum_{i=0}^{p} a_i^2 < \infty, \tag{3.2}$$

where $W_t^{(1)}$ and $W_t^{(2)}$ are two independent standard Brownian processes and $E_i(f_t)$ are the eigenfunctions of the infinitesimal generator of f_t characterized by

$$df_t = \mu(f_t) + \sigma(f_t) dW_t^{(2)}. \tag{3.3}$$

[5]See Chen, Hansen and Scheinkman (2000) for a general approach of eigenfunction modeling in the multivariate case.

3.1 Unconditional moments

In this section we will compute the first four moments of the returns as well as the variance, skewness and kurtosis. We recall that, for a given random variable z, with finite fourth moment, the later are defined by

$$Var[z] = E[(z - E[z])^2]; \; Skew[z] = \frac{E[(z - E[z])^3]}{(Var[z])^{3/2}}; \; Kurt[z] = \frac{E[(z - E[z])^4]}{(Var[z])^2}. \tag{3.4}$$

Proposition 3.1 Marginal moments
Consider $\{y_t, t \in I\!R\}$ a continuous-time ESV(p) and define the returns $\{r_{th}^{(h)}, t \in \mathbf{N}\}$ by

$$r_{th}^{(h)} \equiv y_{th} - y_{(t-1)h}. \tag{3.5}$$

Then the first four moments of $r_{th}^{(h)}$ are given by:

$$E[r_{th}^{(h)}] = 0; \; E[r_{th}^{(h)2}] = a_0 h; \; E[r_{th}^{(h)3}] = 0; \tag{3.6}$$

$$E[r_{th}^{(h)4}] = 3a_0^2 h^2 + 6 \sum_{i=1}^{p} \frac{a_i^2}{\delta_i^2}[-1 + \delta_i h + \exp(-\delta_i h)]. \tag{3.7}$$

As a consequence, the variance, skewness and kurtosis of the returns are:

$$Var[r_{th}^{(h)}] = a_0 h; Skew[r_{th}^{(h)}] = 0; Kurt[r_{th}^{(h)}] = 3 + \frac{6}{a_0^2 h^2} \sum_{i=1}^{p} \frac{a_i^2}{\delta_i^2}[-1 + \delta_i h + \exp(-\delta_i h)]. \tag{3.8}$$

Corollary 3.1 *a- When $h \to 0$, we have*

$$Kurt[r_{th}^{(h)}] \to 3 + 3\frac{\sum_{i=1}^{p} a_i^2}{a_0^2}.$$

b- When $h \to \infty$, we have

$$Kurt[r_{th}^{(h)}] \to 3$$

and

$$Kurt[r_{th}^{(h)}] - 3 \sim \frac{3}{a_0^2} \sum_{i=1}^{p} \frac{a_i^2}{\delta_i} \quad \frac{1}{h}.$$

3.2 Covariance structures

We now compute the covariance structure of the returns, their square and the cross-covariance between the returns and their square.

Proposition 3.2 Covariance structures
Consider $\{y_t, t \in I\!R\}$ *a continuous-time ESV(p) and define* $\{r^{(h)}, t \in \mathbf{N}\}$ *by (3.5). Then, we have:*

$$\forall j \neq 0,\ Cov[r_{th}^{(h)}, r_{(t-j)h}^{(h)}] = 0; \tag{3.9}$$

$$\forall j > 0,\ Cov[r_{th}^{(h)2}, r_{(t-j)h}^{(h)2}] = \sum_{i=1}^{p} \frac{a_i^2}{\delta_i^2}[1 - \exp(-\delta_i h)]^2 \exp(-\delta_i (j-1)h). \tag{3.10}$$

$$\forall j \neq 0,\ Cov[r_{th}^{(h)2}, r_{(t-j)h}^{(h)}] = 0. \tag{3.11}$$

Corollary 3.2 *Let* $Corr[r_{th}^{(h)2}, r_{(t-j)h}^{(h)2}]$ *be the correlation between* $r_{th}^{(h)2}$ *and* $r_{(t-j)h}^{(h)2}$, *i.e.*

$$Corr[r_{th}^{(h)2}, r_{(t-j)h}^{(h)2}] = \frac{Cov[r_{th}^{(h)2}, r_{(t-j)h}^{(h)2}]}{Var[r_{th}^{(h)2}]}.$$

a- When $h \to 0$, *we have*

$$Corr[r_{th}^{(h)2}, r_{(t-j)h}^{(h)2}] \to \frac{\sum_{i=1}^{p} a_i^2}{2a_0^2 + 3\sum_{i=1}^{p} a_i^2} \leq \frac{1}{3}.$$

b- When $h \to \infty$, *we have*

$$Corr[r_{th}^{(h)2}, r_{(t-j)h}^{(h)2}] \to 0,$$

$$\text{for } j = 1,\ Corr[r_{th}^{(h)2}, r_{(t-1)h}^{(h)2}] \sim \frac{\sum_{i=1}^{p} a_i^2/\delta_i^2}{2a_0^2} \frac{1}{h^2},$$

$$\text{for } j > 1,\ Corr[r_{th}^{(h)2}, r_{(t-j)h}^{(h)2}] = o(\frac{1}{h^n}), \quad \forall n.$$

3.3 Conditional moments

We now compute the conditional moments of the returns. Let J_t be the filtration defined by

$$J_t = \sigma(y_\tau, f_\tau, \tau \in I\!R, \tau \leq t) = \sigma(dW_\tau^{(1)}, dW_\tau^{(2)}, \tau \in I\!R, \tau \leq t). \tag{3.12}$$

Proposition 3.3 Conditional moments
Consider $\{y_t, t \in I\!R\}$ a continuous-time ESV(p) and define $\{r^{(h)}, t \in \mathbf{N}\}$ by (3.5). Then, we have:

$$E[r_{th}^{(h)} \mid J_{(t-1)h}] = 0, \tag{3.13}$$

$$E[r_{th}^{(h)2} \mid J_{(t-1)h}] = a_0 h + \sum_{i=1}^{p} \frac{a_i}{\delta_i}[1 - \exp(-\delta_i h)] E_i(f_{(t-1)h}), \tag{3.14}$$

$$E[r_{th}^{(h)3} \mid J_{(t-1)h}] = 0, \tag{3.15}$$

$$E[r_{th}^{(h)4} \mid J_{(t-1)h}] = 3a_0^2 h^2 + 6a_0 \sum_{i=1}^{p} \frac{a_i}{\delta_i^2}[\exp(-\delta_i h) - 1 + \delta_i h] E_i(f_{(t-1)h}) + 6 \sum_{k=0}^{\bar{\bar{p}}} l_k(h) E_k(f_{(t-1)h}) \tag{3.16}$$

with

$$l_k(h) = \sum_{i=1}^{p} a_i \left(\sum_{j=0}^{p} \frac{a_j w_{i,j,k}}{\delta_i - \delta_k} \left[\frac{1 - \exp(-\delta_k h)}{\delta_k} - \frac{1 - \exp(-\delta_i h)}{\delta_i} \right] \right), \tag{3.17}$$

$\bar{\bar{p}} = \sup\{\bar{p}_i, 1 \leq i\}$, *where* $\bar{p}_i = \sup\{p_{i,j}, 1 \leq j\}$, *and* $w_{i,j,k}$ *and* $p_{i,j}$ *are given in (2.21).*

In the previous proposition, the conditional moments where derived given the state variable f_t which is not always observable by the econometrician. We now derive conditional moments given the information of the econometrician.

Proposition 3.4 Conditional moments based on observable variables
Consider $\{y_t, t \in I\!R\}$ a continuous-time ESV(p) and define $\{r^{(h)}, t \in \mathbf{N}\}$ by (3.5). Assume that $p < +\infty$, then we have:

$$E[\prod_{i=1}^{p}(1 - \exp(-\delta_i h) L)[r_{th}^{(h)2} - a_0 h] \mid r_{\tau h}^{(h)}, z_{\tau h}, \tau \leq t - p - 1] = 0, \tag{3.18}$$

where $z_{\tau h}$ are any observarble variables. As a consequence, $r_{th}^{(h)2}$ is an ARMA(p,p) model with autoregressive coefficients $\exp(-\delta_i h)$*,* $i = 1, 2, ..., p$*.*

Assume that $\bar{\bar{p}} < +\infty$, then we have:

$$E[\prod_{i=1}^{\bar{\bar{p}}}(1 - \exp(-\delta_i h)L)[r_{th}^{(h)4} - 2a_0^2 h^2 - l_0(h)] \mid r_{\tau h}^{(h)}, z_{\tau h}, \tau \leq t - \bar{\bar{p}} - 1] = 0. \quad (3.19)$$

As a consequence, $r_{th}^{(h)4}$ is an ARMA($\bar{\bar{p}}$,$\bar{\bar{p}}$) model with autoregressive coefficients $\exp(-\delta_i h)$*,* $i = 1, 2, ..., \bar{\bar{p}}$*.*

Typically, the instrumental variable z_t may be Black-Scholes implied volatilities. Observe that we do not need to specify the price of risk. Other possible instruments are high frequency realized volatilities, range variables, volume of transactions, etc.

4 Incorporating leverage effect

In this section, we incorporate in the model considered in the previous section a constant drift and a leverage effect, i.e.:

$$dy_t = mdt + \sqrt{\sigma_t^2}\ \left[\sqrt{1-\rho^2} dW_t^{(1)} + \rho dW_t^{(2)}\right], \text{ with} \quad (4.1)$$

$$\sigma_t^2 = \sum_{i=0}^{p} a_i E_i(f_t), \qquad p \in \mathbf{N} \bigcup \{+\infty\}, \qquad \sum_{i=0}^{p} a_i^2 < \infty, \quad (4.2)$$

where $W_t^{(1)}$ and $W_t^{(2)}$ are two independent standard Brownian processes and $E_i(f_t)$ are the eigenfunctions of the infinitesimal generator of f_t characterized by

$$df_t = \mu(f_t) + \sigma(f_t) dW_t^{(2)}. \quad (4.3)$$

4.1 Unconditional moments

In this section we will compute the first four moments of the returns as well as the variance, skewness and kurtosis.

Proposition 4.1 Marginal moments
Consider $\{y_t, t \in I\!R\}$ a continuous-time ESV(p) and define the returns $\{r_{th}^{(h)}, t \in \mathbf{N}\}$ by

$$r_{th}^{(h)} \equiv y_{th} - y_{(t-1)h}. \quad (4.4)$$

Then the first four moments of $r_{th}^{(h)}$ *are given by:*

$$E[r_{th}^{(h)}] = mh; \;\; E[r_{th}^{(h)2}] = m^2h^2 + a_0h; \tag{4.5}$$

$$E[r_{th}^{(h)3}] = m^3h^3 + 3ma_0h^2 + 3\rho\sum_{i=1}^{p}\frac{ae_{i,0}}{\delta_i^2}[\exp(-\delta_i h) - 1 + \delta_i h]; \tag{4.6}$$

$$E[r_{th}^{(h)4}] = m^4h^4 + 6m^2a_0h^3 + 3a_0^2h^26\sum_{i=1}^{p}\frac{a_i^2}{\delta_i^2}[-1 + \delta_i h + \exp(-\delta_i h)]$$

$$+12m\rho h\sum_{i=1}^{p}\frac{ae_{i,0}}{\delta_i^2}[\exp(-\delta_i h) - 1 + \delta_i h] + 12\rho^2 n_0(h) \tag{4.7}$$

$$\textit{where } n_k(h) = \sum_{i=1}^{p} a_i\left(\sum_{j=0}^{p_i}\frac{e_{i,j}e_{j,k}}{\delta_j - \delta_k}\left[\frac{1-\exp(-\delta_k h)}{(\delta_i - \delta_k)\delta_k} - \frac{1-\exp(-\delta_i h)}{(\delta_i - \delta_k)\delta_i}\right.\right.$$
$$\left.\left.- \frac{1-\exp(-\delta_j h)}{(\delta_i - \delta_j)\delta_j} + \frac{1-\exp(-\delta_i h)}{(\delta_i - \delta_j)\delta_i}\right]\right). \tag{4.8}$$

As a consequence, the variance, skewness and kurtosis of the returns are:

$$Var[r_{th}^{(h)}] = a_0h; \tag{4.9}$$

$$Skew[r_{th}^{(h)}] = 3\rho\sum_{i=1}^{p}\frac{ae_{i,0}}{\delta_i^2}[\exp(-\delta_i h) - 1 + \delta_i h], \tag{4.10}$$

$$Kurt[r_{th}^{(h)}] = 3 + \frac{6}{a_0^2h^2}\sum_{i=1}^{p}\frac{a_i^2}{\delta_i^2}[-1 + \delta_i h + \exp(-\delta_i h)] + \frac{12\rho^2}{a_0^2h^2}n_0(h) \tag{4.11}$$

4.2 Conditional moments

We now compute the conditional moments of the returns. Let J_t be the filtration defined by

$$J_t = \sigma(y_\tau, f_\tau, \tau \in I\!R, \tau \le t) = \sigma(dW_\tau^{(1)}, dW_\tau^{(2)}, \tau \in I\!R, \tau \le t). \tag{4.12}$$

Proposition 4.2 Conditional moments

Consider $\{y_t, t \in I\!R\}$ a continuous-time ESV(p) and define $\{r^{(h)}, t \in \mathbf{N}\}$ by (4.4). Then, we have:

$$E[r_{th}^{(h)} \mid J_{(t-1)h}] = mh, \tag{4.13}$$

$$E[r_{th}^{(h)2} \mid J_{(t-1)h}] = m^2h^2 + a_0h + \sum_{i=1}^{p} \frac{a_i}{\delta_i}[1 - \exp(-\delta_i h)]E_i(f_{(t-1)h}), \tag{4.14}$$

$$\begin{aligned} E[r_{th}^{(h)3} \mid J_{(t-1)h}] = m^3h^3 + 3ma_0h^2 + 3m\sum_{i=1}^{p} \frac{a_ih}{\delta_i}(1 - \exp(-\delta_i h))E_i(f_{(t-1)h}) \\ +3\rho \sum_{j=0}^{\bar{p}} d_j(h)E_j(f_{(t-1)h}) \end{aligned} \tag{4.15}$$

where

$$\bar{p} \equiv \sup\{p_i, 1 \leq i \leq p\} \quad \text{and} \quad d_j(h) = \sum_{i=1}^{p} \frac{a_ie_{i,j}}{\delta_i - \delta_j}[\frac{1 - \exp(-\delta_j h)}{\delta_j} - \frac{1 - \exp(-\delta_i h)}{\delta_i}]. \tag{4.16}$$

$$\begin{aligned} E[r_{th}^{(h)4} \mid J_{(t-1)h}] = m^4h^4 + 6m^2a_0h^3 + 6m^2h^2\sum_{i=1}^{p} \frac{a_i}{\delta_i}(1 - \exp(-\delta_i h))E_i(f_{(t-1)h}) \\ +3a_0^2h^2 + 6a_0\sum_{i=1}^{p} \frac{a_i}{\delta_i^2}[\exp(-\delta_i h) - 1 + \delta_i h]E_i(f_{(t-1)h}) \\ +6\sum_{k=0}^{\bar{\bar{p}}} l_k(h)E_k(f_0) + 12m\rho h\sum_{j=0}^{\bar{p}} m_j(h)E_j(f_{(t-1)h}) + 12\rho^2\sum_{k=0}^{\bar{p}} n_k(h)E_k(f_{(t-1)h}) \end{aligned} \tag{4.17}$$

where

$$\bar{\bar{p}} = \sup\{\bar{p}_i, 1 \leq i\}, \quad l_k(h) = \sum_{i=1}^{p} a_i \left(\sum_{j=0}^{p} \frac{a_jw_{i,j,k}}{\delta_i - \delta_k}\left[\frac{1 - \exp(-\delta_k h)}{\delta_k} - \frac{1 - \exp(-\delta_i h)}{\delta_i}\right]\right), \tag{4.18}$$

$$m_j(h) = \sum_{i=1}^{p} \frac{a_ie_{i,j}}{\delta_i - \delta_j}\left[\frac{1 - \exp(-\delta_j h)}{\delta_j} - \frac{1 - \exp(-\delta_i h)}{\delta_i}\right], \tag{4.19}$$

$$
n_k(h) = \sum_{i=1}^{p} a_i \left(\sum_{j=0}^{p_i} \frac{e_{i,j} e_{j,k}}{\delta_j - \delta_k} \left[\frac{1-\exp(-\delta_k h)}{(\delta_i - \delta_k)\delta_k} - \frac{1-\exp(-\delta_i h)}{(\delta_i - \delta_k)\delta_i} \right.\right.
$$
$$
\left.\left. - \frac{1-\exp(-\delta_j h)}{(\delta_i - \delta_j)\delta_j} + \frac{1-\exp(-\delta_i h)}{(\delta_i - \delta_j)\delta_i} \right] \right). \tag{4.20}
$$

In the previous proposition, the conditional moments where derived given the state variable f_t which is not always observable by the econometrician. As is in the previous section, it is possible to derive conditional moments given the information of the econometrician.

5 Conclusion

In this paper, we derive explicit formulas for conditional and unconditional moments of the continuous time Eigenfunction Stochastic Volatility (ESV) models of Meddahi (2001). Special cases of ESV models are log-normal, affine and GARCH diffusion models. The conditional moment restrictions we derive are based only on observable variables. Therefore, using an instrumental variable approach is easy. A major advantage of this approach with respect to other methods (e.g., simulation techniques) is that one can use extra variables as instruments, in particular Black-Scholes implied volatilities (without specifying a price of risk) and high-frequency realized volatility. Moment conditions are also useful for statistical inference purposes. This is of interest given that we observe many financial derivative prices. Therefore, one can test which features of the model are or are not compatible with financial derivatives.

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Appendix

The propositions given in the text are special examples of the following lemmas.

In the sequel, we will consider the process Z_t defined by

$$Z_t = \int_0^t dy_u = mt + \int_0^t \sqrt{\sigma_u^2}[\sqrt{1-\rho^2}dW_u^{(1)} + \rho dW_u^{(2)}]. \tag{A.1}$$

and compute various conditional and unconditional moments of Z_h where h is a real number.

Before doing so, let us recall the Ito's Formula (e.g., see Protter, 1995, page 74). Consider X a continuous semimartingale and let f a $\mathcal{C}^2$ function, then

$$f(X_h) - f(X_0) = \int_{0+}^h f'(X_u)dX_u + \frac{1}{2}\int_{0+}^h f''(X_u)d[X,X]_u \tag{A.2}$$

where $[X,X]_s$ is the quadratic variation of X_s (Protter, 1995, page 58). In the following, we will use Ito's formula to compute any moment of order r of Z_h by considering the function $f(x) = x^r$. Observe that

$$dZ_u = mdu + \sqrt{\sigma_u^2}[\sqrt{1-\rho^2}dW_u^{(1)} + \rho dW_u^{(2)}], \tag{A.3}$$

$$[Z,Z]_u = \int_0^u \sigma_s^2 ds \quad \text{and} \quad d[Z,Z]_u = \sigma_u^2 du. \tag{A.4}$$

Lemma A1: Conditional and unconditional first and second moments. *Let $h > 0$ and consider Z_h defined in (A.1) and I_0 the information at time 0. Then we have*

$$E[Z_h \mid I_0] = mh \quad \text{and} \tag{A.5}$$

$$E[Z_h^2 \mid I_0] = m^2h^2 + a_0h + \sum_{i=1}^p \frac{a_i}{\delta_i}(1-\exp(-\delta_i h))E_i(f_0). \tag{A.6}$$

Hence,

$$E[Z_h] = mh \quad \text{and} \quad E[Z_h^2] = m^2h^2 + a_0h. \tag{A.7}$$

Proof. We have: $E[\int_0^h \sqrt{\sigma_u^2}[\sqrt{1-\rho^2}dW_u^{(1)} + \rho dW_u^{(2)}] \mid I_0] = 0$. Hence, $E[Z_h \mid I_0] = mh$ and $E[Z_h] = mh$.

By using Ito's formula (A.2), we get:

$Z_h^2 = 2\int_0^h Z_u dZ_u + \int_0^h d[Z,Z]_u$
$\quad = 2m\int_0^h Z_u du + 2\int_0^h Z_u \sqrt{\sigma_u^2} \ \left[\sqrt{1-\rho^2}dW_u^{(1)} + \rho dW_u^{(2)}\right] + \int_0^h \sigma_u^2 du.$

Therefore,

$$\begin{aligned} E[Z_h^2 \mid I_0] &= 2m \int_0^h E[Z_u \mid I_0]du + 0 + \int_0^h E[\sigma_u^2 \mid I_0]du \\ &= 2m \int_0^h mudu + \sum_{i=0}^{p} a_i \int_0^h \exp(-\delta_i u)du \ E_i(f_0) \\ &= m^2h^2 + a_0 h + \sum_{i=1}^{p} \frac{a_i}{\delta_i}(1-\exp(-\delta_i h)) \ E_i(f_0). \end{aligned}$$

As a consequence, $E[Z_h^2] = m^2h^2 + a_0 h$.■

Lemma A2: *We have $E[Z_u E_0(f_u) \mid I_0] = mu$ and for $i \neq 0$:*

$$E[Z_u E_i(f_u) \mid I_0] = mu\exp(-\delta_i u)E_i(f_0) + \rho \sum_{j=0}^{p_i} e_{i,j} \frac{\exp(-\delta_j u) - \exp(-\delta_i u)}{\delta_i - \delta_j} E_j(f_0). \tag{A.8}$$

Proof. We have $E[Z_u E_0(f_u) \mid I_0] = E[Z_u \mid I_0] = mu$. For $i \neq 0$, we have:

$$\begin{aligned} E[Z_u E_i(f_u) \mid I_0] &= E[\Big(mu + \textstyle\int_0^u \sqrt{\sigma_s^2} \ \left[\sqrt{1-\rho^2}dW_s^{(1)} + \rho dW_s^{(2)}\right]\Big) E_i(f_u) \mid I_0] \\ &= muE[E_i(f_u) \mid I_0] + \rho \textstyle\int_0^u E[\sqrt{\sigma_s^2}dW_s^{(2)}E_i(f_u) \mid I_0] \\ &= mu\exp(-\delta_i u)E_i(f_0) + \rho \textstyle\int_0^u E[E[\sqrt{\sigma_s^2}dW_s^{(2)}E_i(f_u) \mid I_s] \mid I_0] \\ &= mu\exp(-\delta_i u)E_i(f_0) + \rho \textstyle\int_0^u E[\exp(-\delta_i(u-s))\sqrt{\sigma_s^2}\sigma(f_s)E_i'(f_s) \mid I_0]ds \\ &= mu\exp(-\delta_i u)E_i(f_0) + \rho \textstyle\int_0^u E[\exp(-\delta_i(u-s)) \sum_{j=0}^{p_i} e_{i,j}E_j(f_s) \mid I_0]ds \\ &= mu\exp(-\delta_i u)E_i(f_0) + \rho \textstyle\sum_{j=0}^{p_i} e_{i,j}(\int_0^u \exp(-\delta_i(u-s))\exp(-\delta_j s)ds) \ E_j(f_0) \\ &= mu\exp(-\delta_i u)E_i(f_0) + \rho \sum_{j=0}^{p_i} e_{i,j} \frac{\exp(-\delta_j u) - \exp(-\delta_i u)}{\delta_i - \delta_j} E_j(f_0). \blacksquare \end{aligned}$$

Lemma A3: Conditional and unconditional third moment. *Let $h > 0$ and consider Z_h defined in (A.1) and I_0 the information at time 0. Then we have*

$$E[Z_h^3 \mid I_0] = m^3h^3 + 3ma_0h^2 + 3m\sum_{i=1}^{p} \frac{a_i h}{\delta_i}(1-\exp(-\delta_i h))E_i(f_0) + 3\rho \sum_{j=0}^{\bar{p}} d_j(h)E_j(f_0) \tag{A.9}$$

where

$$\bar{p} \equiv \sup\{p_i, 1 \le i \le p\} \quad \text{and} \quad d_j(h) = \sum_{i=1}^{p} \frac{a_i e_{i,j}}{\delta_i - \delta_j}[\frac{1-\exp(-\delta_j h)}{\delta_j} - \frac{1-\exp(-\delta_i h)}{\delta_i}]. \tag{A.10}$$

Hence,

$$E[Z_h^3] = m^3h^3 + 3ma_0h^2 + 3\rho\sum_{i=1}^{p}\frac{a_i e_{i,0}}{\delta_i}[h - \frac{1-\exp(-\delta_i h)}{\delta_i}]. \tag{A.11}$$

Proof. By using Ito's Lemma, we get:

$Z_h^3 = 3\int_0^h Z_u^3 dZ_u + 3\int_0^h Z_u d[Z,Z]_u$

$\quad = 3m\int_0^h Z_u^2 du + 3\int_0^h Z_u^2\sqrt{\sigma_u^2}\ \left[\sqrt{1-\rho^2}dW_u^{(1)} + \rho dW_u^{(2)}\right] + 3\int_0^h Z_u\sigma_u^2 du.$

Therefore, $E[Z_h^3 \mid I_0] = 3m\int_0^h E[Z_u^2 \mid I_0]du + 3\int_0^h E[Z_u\sigma_u^2 \mid I_0]du.$

Let us first compute the first term. We have:

$$3m\int_0^h E[Z_u^2 \mid I_0]du = 3m\int_0^h \left(m^2u^2 + a_0u + \sum_{i=1}^{p}\frac{a_i}{\delta_i}(1-\exp(-\delta_i u))E_i(f_0)\right)du$$

$$= m^3h^3 + 3ma_0\frac{h^2}{2} + 3m\sum_{i=1}^{p}\frac{a_i}{\delta_i^2}(\delta_i h - 1 + \exp(-\delta_i h))E_i(f_0).$$

Now we consider the second term. We have:

$E[Z_u\sigma_u^2 \mid I_0] = \sum_{i=0}^{p} a_i E[Z_u E_i(f_u) \mid I_0]$

$$= a_0mu + \sum_{i=1}^{p} a_i\left(mu\exp(-\delta_i u)E_i(f_0) + \rho\sum_{j=0}^{p_i} e_{i,j}\frac{\exp(-\delta_j u) - \exp(-\delta_i u)}{\delta_i - \delta_j}E_j(f_0)\right).$$

Hence,

$$\int_0^h E[Z_u\sigma_u^2 \mid I_0]du = a_0m\frac{h^2}{2} + \sum_{i=1}^{p}\frac{a_i}{\delta_i^2}[1 - \delta_i h\exp(-\delta_i h) - \exp(-\delta_i h)]E_i(f_0)$$
$$+ \rho\sum_{i=1}^{p} a_i\sum_{j=0}^{p_i}\frac{e_{i,j}}{\delta_i - \delta_j}[\frac{1-\exp(-\delta_j h)}{\delta_j} - \frac{1-\exp(-\delta_i h)}{\delta_i}]E_j(f_0).$$

As a consequence,

$$E[Z_h^3 \mid I_0] = m^3h^3 + 3ma_0\frac{h^2}{2} + 3m\sum_{i=1}^{p}\frac{a_i}{\delta_i^2}(\delta_i h - 1 + \exp(-\delta_i h))E_i(f_0)$$
$$+3ma_0\frac{h^2}{2} + 3m\sum_{i=1}^{p}\frac{a_i}{\delta_i^2}[1 - \delta_i h\exp(-\delta_i h) - \exp(-\delta_i h)]E_i(f_0)$$
$$+3\rho\sum_{i=1}^{p} a_i\sum_{j=0}^{p_i}\frac{e_{i,j}}{\delta_i - \delta_j}[\frac{1-\exp(-\delta_j h)}{\delta_j} - \frac{1-\exp(-\delta_i h)}{\delta_i}]E_j(f_0)$$
$$= m^3h^3 + 3ma_0h^2 + 3m\sum_{i=1}^{p}\frac{a_i h}{\delta_i}(1-\exp(-\delta_i h))E_i(f_0)$$

$$+3\rho\sum_{i=1}^{p} a_i \sum_{j=0}^{p_i} \frac{e_{i,j}}{\delta_i-\delta_j}[\frac{1-\exp(-\delta_j h)}{\delta_j}-\frac{1-\exp(-\delta_i h)}{\delta_i}]E_j(f_0).$$

By arranging the terms in the previous equation, one gets (A.9). Therefore,

$$E[Z_h^3] = m^3h^3 + 3ma_0h^2 + 3\rho\sum_{i=1}^{p} a_i \frac{e_{i,0}}{\delta_i-\delta_0}[\frac{1-\exp(-\delta_0 h)}{\delta_0}-\frac{1-\exp(-\delta_i h)}{\delta_i}].$$

But $\delta_0 = 0$ and $\dfrac{1-\exp(-\delta_0 h)}{\delta_0} = h$. Hence we get (A.11). ■

Lemma A4: *For $i \neq 0$ we have:*

$$E[Z_h^2 E_i(f_h) \mid I_0] = m^2h^2\exp(-\delta_i h)E_i(f_0) + 2m\rho h \sum_{j=0}^{p_i} \frac{e_{i,j}}{(\delta_i-\delta_j)}(\exp(-\delta_j h)-\exp(-\delta_i h))E_j(f_0)$$

$$+\sum_{k=0}^{\bar{p}_i} s_{i,k}(h)E_k(f_0) + 2\rho^2\sum_{k=0}^{\bar{p}} \bar{d}_{i,k}(h)E_k(f_0) \tag{A.12}$$

where

$$\bar{p}_i \equiv \sup\{p_{i,j}, 0 \leq j \leq p\}, \quad s_{i,k}(h) = \sum_{j=0}^{p} a_j w_{i,j,k}\frac{\exp(-\delta_k h)-\exp(-\delta_i h)}{\delta_i-\delta_k} \tag{A.13}$$

and

$$d_{i,k}(h) = \sum_{j=0}^{p_i} \frac{e_{i,j}e_{j,k}}{\delta_j-\delta_k}[\frac{\exp(-\delta_k h)-\exp(-\delta_i h)}{\delta_i-\delta_k}-\frac{\exp(-\delta_j h)-\exp(-\delta_i h)}{\delta_i-\delta_j}]. \tag{A.14}$$

Proof. Let $i \neq 0$. We have

$E[Z_h^2 E_i(f_h) \mid I_0]$

$= E\left[\left(2m\int_0^h Z_u du + \int_0^h \sigma_u^2 du + 2\int_0^h Z_u\sqrt{\sigma_u^2}\ \left[\sqrt{1-\rho^2}dW_u^{(1)} + \rho dW_u^{(2)}\right]\right)E_i(f_h) \mid I_0\right]$

We have to compute the there terms.

1) $E[\int_0^h Z_u du \ \ E_i(f_h) \mid I_0] = E[\int_0^h Z_u E[E_i(f_h) \mid I_u]du \mid I_0]$

$= E[\int_0^h Z_u \exp(-\delta_i(h-u))E_i(f_u)du \mid I_0]$

$= \int_0^h \exp(-\delta_i(h-u))E[Z_u E_i(f_u) \mid I_0]du$

$= \int_0^h \exp(-\delta_i(h-u))\left(mu\exp(-\delta_i u)E_i(f_0) + \rho\sum_{j=0}^{p_i} e_{i,j}\frac{\exp(-\delta_j u)-\exp(-\delta_i u)}{\delta_i-\delta_j}E_j(f_0)\right)du$

$= \exp(-\delta_i h)\left(m\frac{h^2}{2}E_i(f_0) + \rho\sum_{j=0}^{p_i}\frac{e_{i,j}}{(\delta_i-\delta_j)^2}[\exp((\delta_i-\delta_j)h) - 1 - h(\delta_i-\delta_j)]E_j(f_0)\right).$

2) $E[\int_0^h \sigma_u^2 du \ E_i(f_h) \mid I_0] = E[\int_0^h \exp(-\delta_i(h-u))E_i(f_u)\sigma_u^2 du \mid I_0]$

$$= \sum_{j=0}^{p} a_j \int_0^h \exp(-\delta_i(h-u))E[E_i(f_u)f_j(f_u) \mid I_0]du$$
$$= \sum_{j=0}^{p} a_j \int_0^h \exp(-\delta_i(h-u))E[\sum_{k=0}^{p_{i,j}} w_{i,j,k}E_k(f_u) \mid I_0]du$$
$$= \sum_{j=0}^{p} a_j \sum_{k=0}^{p_{i,j}} w_{i,j,k} \int_0^h \exp(-\delta_i(h-u))\exp(-\delta_k u)du \; E_k(f_0)$$
$$= \sum_{j=0}^{p} a_j \sum_{k=0}^{p_{i,j}} w_{i,j,k} \frac{\exp(-\delta_k h) - \exp(-\delta_i h)}{\delta_i - \delta_k} \; E_k(f_0).$$

3) Since the $dW^{(1)}$ and $dW^{(2)}$ are independent, we have

$$E[\int_0^h Z_u \sqrt{\sigma_u^2}\sqrt{1-\rho^2}dW_u^{(1)}E_i(f_h) \mid I_0]$$
$$= E[\int_0^h Z_u \sqrt{\sigma_u^2}\sqrt{1-\rho^2}\exp(-\delta_i(u-h))E_i(f_h)dW_u^{(1)} \mid I_0] = 0.$$

Let us compute the second term. We have

$$E[\int_0^h Z_u \sqrt{\sigma_u^2}dW_u^{(2)}E_i(f_h) \mid I_0] = E[\int_0^h Z_u \sqrt{\sigma_u^2}\exp(-\delta_i(h-u))\sigma(f_u)E_i'(f_u)du \mid I_0]$$
$$= E[\int_0^h Z_u \exp(-\delta_i(h-u))(\sum_{j=0}^{p_i} e_{i,j}E_j(f_u))du \mid I_0]$$
$$= E[\int_0^h \left(mu + \int_0^u \sqrt{\sigma_s^2}(\rho dW_s^{(2)} + \sqrt{1-\rho^2}dW_s^{(1)})\right)\exp(-\delta_i(h-u))(\sum_{j=0}^{p_i} e_{i,j}E_j(f_u))du \mid I_0]$$
$$= E[\int_0^h \left(mu + \rho\int_0^u \sqrt{\sigma_s^2}dW_s^{(2)}\right)\exp(-\delta_i(h-u))(\sum_{j=0}^{p_i} e_{i,j}E_j(f_u))du \mid I_0].$$

$$E[\int_0^h mu\exp(-\delta_i(h-u))(\sum_{j=0}^{p_i} e_{i,j}E_j(f_u))du \mid I_0]$$
$$= m\sum_{j=0}^{p_i} e_{i,j}\exp(-\delta_i h)\int_0^h u\exp((\delta_i-\delta_j)u)du \; E_j(f_0)$$
$$= m\sum_{j=0}^{p_i} \frac{e_{i,j}}{(\delta_i-\delta_j)^2}[\exp(-\delta_j h)((\delta_i-\delta_j)h-1)+\exp(-\delta_i h)]E_j(f_0).$$

$$E[\int_0^h (\int_0^u \sqrt{\sigma_s^2}dW_s^{(2)})\exp(-\delta_i(h-u))(\sum_{j=0}^{p_i} e_{i,j}E_j(f_u))du \mid I_0]$$
$$= \sum_{j=0}^{p_i} e_{i,j}E[\int_0^h (\int_0^u \sqrt{\sigma_s^2}dW_s^{(2)})\exp(-\delta_i(h-u))E_j(f_u))du \mid I_0]$$
$$= \sum_{j=0}^{p_i} e_{i,j}E[\int_0^h (\int_0^u \sqrt{\sigma_s^2}dW_s^{(2)})\exp(-\delta_i(h-u))E_j(f_u))du \mid I_0]$$

$$= \sum_{j=0}^{p_i} e_{i,j} E[\int_0^h (\int_0^u \sqrt{\sigma_s^2} \exp(-\delta_j(u-s))\sigma(f_s)E_j'(f_s)ds)\exp(-\delta_i(h-u))du \mid I_0]$$
$$= \sum_{j=0}^{p_i} e_{i,j} E[\int_0^h (\int_0^u \exp(-\delta_j(u-s))(\sum_{k=0}^{p_j} e_{j,k}E_k(f_s))ds)\exp(-\delta_i(h-u))du \mid I_0]$$
$$= \sum_{j=0}^{p_i} e_{i,j} \int_0^h (\int_0^u \exp(-\delta_j(u-s))(\sum_{k=0}^{p_j} e_{j,k}\exp(-\delta_k s)E_k(f_0))ds)\exp(-\delta_i(h-u))du$$
$$= \sum_{j=0}^{p_i} e_{i,j} \sum_{k=0}^{p_j} e_{j,k} \int_0^h \int_0^u \exp(-\delta_j(u-s))(\exp(-\delta_k s)ds \exp(-\delta_i(h-u))du\ E_k(f_0))$$
$$= \sum_{j=0}^{p_i} e_{i,j} \sum_{k=0}^{p_j} e_{j,k} \exp(-\delta_i h) \int_0^h \exp((\delta_i-\delta_j)u)(\int_0^u \exp((\delta_j-\delta_k)s)ds)du\ E_k(f_0))$$
$$= \sum_{j=0}^{p_i} e_{i,j} \sum_{k=0}^{p_j} e_{j,k} \frac{1}{\delta_j-\delta_k}[\frac{\exp(-\delta_k h)-\exp(-\delta_i h)}{\delta_i-\delta_k} - \frac{\exp(-\delta_j h)-\exp(-\delta_i h)}{\delta_i-\delta_j}]E_k(f_0).$$

As a summary, we have for $i \neq 0$:

$$E[Z_h^2 E_i(f_h) \mid I_0] = m^2h^2\exp(-\delta_i h)E_i(f_0)$$
$$+2m\rho \sum_{j=0}^{p_i} \frac{e_{i,j}}{(\delta_i-\delta_j)^2}[\exp((\delta_i-\delta_j)h) - 1 - h(\delta_i-\delta_j)]E_j(f_0)$$
$$+\sum_{j=0}^{p} a_j \sum_{k=0}^{p_{i,j}} w_{i,j,k} \frac{\exp(-\delta_k h)-\exp(-\delta_i h)}{\delta_i-\delta_k}\ E_k(f_0)$$
$$+2\rho m \sum_{j=0}^{p_i} \frac{e_{i,j}}{(\delta_i-\delta_j)^2}[\exp(-\delta_j h)((\delta_i-\delta_j)h-1) + \exp(-\delta_i h)]E_j(f_0)$$
$$+2\rho^2 \sum_{j=0}^{p_i} e_{i,j} \sum_{k=0}^{p_j} e_{j,k} \frac{1}{\delta_j-\delta_k}[\frac{\exp(-\delta_k h)-\exp(-\delta_i h)}{\delta_i-\delta_k} - \frac{\exp(-\delta_j h)-\exp(-\delta_i h)}{\delta_i-\delta_j}]E_k(f_0).$$

By combining the second and fourth terms in the previous equality, one gets the second term in (A.12). By rearranging the third term in the previous equality, one gets the third term in (A.12). By rearranging the fifth term in the previous equality, one gets the fourth term in (A.12).■

Lemma A5: Conditional and unconditional fourth moment. *Let $h > 0$ and consider Z_h defined in (A.1) and I_0 the information at time 0. Then we have*

$$E[Z_h^4 \mid I_0] = m^4h^4 + 6m^2a_0h^3 + +6m^2h^2 \sum_{i=1}^{p} \frac{a_i}{\delta_i}(1-\exp(-\delta_i h))E_i(f_0)$$
$$+3a_0^2h^2 + 6a_0 \sum_{i=1}^{p} \frac{a_i}{\delta_i^2}[\exp(-\delta_i h) - 1 + \delta_i h]E_i(f_0)$$

$$+6\sum_{k=0}^{\bar{\bar{p}}} l_k(h)E_k(f_0) + 12m\rho h\sum_{j=0}^{\bar{p}} m_j(h)E_j(f_0) + 12\rho^2\sum_{k=0}^{\bar{p}} n_k(h)E_k(f_0) \qquad \text{(A.15)}$$

where

$$\bar{\bar{p}} = \sup\{\bar{p}_i, 1 \leq i \leq 1\}, \quad l_k(h) = \sum_{i=1}^{p} a_i \left(\sum_{j=0}^{p} \frac{a_j w_{i,j,k}}{\delta_i - \delta_k}\left[\frac{1-\exp(-\delta_k h)}{\delta_k} - \frac{1-\exp(-\delta_i h)}{\delta_i}\right]\right), \qquad \text{(A.16)}$$

$$m_j(h) = \sum_{i=1}^{p} \frac{a_i e_{i,j}}{\delta_i - \delta_j}\left[\frac{1-\exp(-\delta_j h)}{\delta_j} - \frac{1-\exp(-\delta_i h)}{\delta_i}\right], \qquad \text{(A.17)}$$

$$n_k(h) = \sum_{i=1}^{p} a_i \left(\sum_{j=0}^{p_i} \frac{e_{i,j}e_{j,k}}{\delta_j - \delta_k}\left[\frac{1-\exp(-\delta_k h)}{(\delta_i - \delta_k)\delta_k} - \frac{1-\exp(-\delta_i h)}{(\delta_i - \delta_k)\delta_i} - \frac{1-\exp(-\delta_j h)}{(\delta_i - \delta_j)\delta_j} + \frac{1-\exp(-\delta_i h)}{(\delta_i - \delta_j)\delta_i}\right]\right). \qquad \text{(A.18)}$$

Hence,

$$E[Z_h^4] = m^4h^4 + 6m^2a_0h^3 + 3a_0^2h^2 + 6l_0(h) + 12m\rho h m_0(h) + 12\rho^2 n_0(h). \qquad \text{(A.19)}$$

Proof. We have:
$Z_h^4 = 4\int_0^h Z_u^3 dZ_u + 6\int_0^h Z_u^2 d[Z,Z]_u$
$= 4m\int_0^h Z_u^3 du + 4\int_0^h Z_u^3\sqrt{\sigma_u^2}[\sqrt{1-\rho^2}dW_u^{(1)} + \rho dW_u^{(2)}] + 6\int_0^h Z_u^2\sigma_u^2 du.$
Hence,
$E[Z_h^4 \mid I_0] = 4m\int_0^h E[Z_u^3 \mid I_0]du + 6\int_0^h E[Z_u^2\sigma_u^2 \mid I_0]du$
$= 4m\int_0^h E[Z_u^3 \mid I_0]du + 6\sum_{i=0}^{p} a_i\int_0^h E[Z_u^2 E_i(f_u) \mid I_0]du.$
We already compute $E[Z_u^3 \mid I_0]$. Moreover, $E[Z_u^2\sigma_u^2 \mid I_0] = \sum_{i=0}^{p} a_i E[Z_u^2 E_i(f_u) \mid I_0]$ which is also already computed. Hence, we have to integrate their expressions to get $E[Z_h^4 \mid I_0]$.
Simple calculus show that

$$\int_0^h u\exp(-\delta_i u)du = \frac{1-\exp(-\delta_i h)(1+\delta_i h)}{\delta_i^2}$$

and

$$\int_0^h u^2 \exp(-\delta_i u)du = \frac{2 - \exp(-\delta_i h)(2 + 2\delta_i h + \delta_i^2 h^2)}{\delta_i^3}.$$

Therefore:

$$\int_0^h E[Z_u^3 \mid I_0] = m^3 \frac{h^4}{4} + m a_0 h^3 + 3m \sum_{i=1}^{p} \frac{a_i}{\delta_i} \left[\frac{h^2}{2} - \frac{1 - \exp(-\delta_i h)(1 + \delta_i h)}{\delta_i^2} \right] E_i(f_0)$$
$$+3\rho \sum_{j=0}^{\bar{p}} \left(\sum_{i=1}^{p} \frac{a_i e_{i,j}}{\delta_i - \delta_j} \left[\frac{\exp(-\delta_j h) - 1 + \delta_j h}{\delta_j^2} - \frac{\exp(-\delta_i h) - 1 + \delta_i h}{\delta_i^2} \right] \right) E_j(f_0).$$

Moreover, for $i \neq 0$:

$$\int_0^h E[Z_u^2 E_i(f_u) \mid I_0] = m^2 \int_0^h \left(u^2 \exp(-\delta_i u) E_i(f_0) \right) du$$
$$+ \int_0^h \left(2m\rho u \sum_{j=0}^{p_i} \frac{e_{i,j}}{(\delta_i - \delta_j)} (\exp(-\delta_j u) - \exp(-\delta_i u)) E_j(f_0) \right) du$$
$$+ \int_0^h \left(\sum_{k=0}^{\bar{p}_i} s_{i,k}(u) E_k(f_0) + 2\rho^2 \sum_{k=0}^{\bar{p}} \bar{d}_{i,k}(u) E_k(f_0) \right) du$$
$$= m^2 \frac{2 - \exp(-\delta_i h)(2 + 2\delta_i h + \delta_i^2 h^2)}{\delta_i^3} E_i(f_0)$$
$$+2m\rho \sum_{j=0}^{p_i} \frac{e_{i,j}}{(\delta_i - \delta_j)} \left[\frac{1 - \exp(-\delta_j h)(1 + \delta_j h)}{\delta_j^2} - \frac{1 - \exp(-\delta_i h)(1 + \delta_i h)}{\delta_i^2} \right] E_j(f_0)$$
$$+ \sum_{k=0}^{\bar{p}_i} \left(\sum_{j=0}^{p} \frac{a_j w_{i,j,k}}{\delta_i - \delta_k} \left[\frac{1 - \exp(-\delta_k h)}{\delta_k} - \frac{1 - \exp(-\delta_i h)}{\delta_i} \right] \right) E_k(f_0)$$
$$+2\rho^2 \sum_{k=0}^{\bar{p}} \left(\sum_{j=0}^{p_i} \frac{e_{i,j} e_{j,k}}{\delta_j - \delta_k} \left[\frac{1 - \exp(-\delta_k h)}{(\delta_i - \delta_k)\delta_k} - \frac{1 - \exp(-\delta_i h)}{(\delta_i - \delta_k)\delta_i} \right.\right.$$
$$\left.\left. - \frac{1 - \exp(-\delta_j h)}{(\delta_i - \delta_j)\delta_j} + \frac{1 - \exp(-\delta_i h)}{(\delta_i - \delta_j)\delta_i} \right] \right) E_k(f_0).$$

For $i = 0$, we have:

$\displaystyle\int_0^h E[Z_u^2 E_0(f_u) \mid I_0] = \int_0^h E[Z_u^2 \mid I_0] = \frac{m^2 h^3}{3} + \frac{a_0 h^2}{2} + \sum_{j=1}^{p} \frac{a_j}{\delta_j^2} [\exp(-\delta_j h) - 1 + \delta_j h] E_j(f_0).$

Therefore,

$$E[Z_h^4 \mid I_0] = m^4 h^4 + 4m^2 a_0 h^3 + 12m^2 \sum_{i=1}^{p} \frac{a_i}{\delta_i} \left[\frac{h^2}{2} - \frac{1 - \exp(-\delta_i h)(1 + \delta_i h)}{\delta_i^2} \right] E_i(f_0)$$
$$+12m\rho \sum_{j=0}^{\bar{p}} \left(\sum_{i=1}^{p} \frac{a_i e_{i,j}}{\delta_i - \delta_j} \left[\frac{\exp(-\delta_j h) - 1 + \delta_j h}{\delta_j^2} - \frac{\exp(-\delta_i h) - 1 + \delta_i h}{\delta_i^2} \right] \right) E_j(f_0)$$
$$+2a_0 m^2 h^3 + 3a_0^2 h^2 + 6a_0 \sum_{i=1}^{p} \frac{a_i}{\delta_i^2} [\exp(-\delta_i h) - 1 + \delta_i h] E_i(f_0)$$

$$+6m^2\sum_{i=1}^{p} a_i \frac{2-\exp(-\delta_i h)(2+2\delta_i h+\delta_i^2 h^2)}{\delta_i^3} E_i(f_0)$$
$$+12m\rho \sum_{i=1}^{p}\sum_{j=0}^{p_i} \frac{e_{i,j}}{(\delta_i-\delta_j)} \left[\frac{1-\exp(-\delta_j h)(1+\delta_j h)}{\delta_j^2} - \frac{1-\exp(-\delta_i h)(1+\delta_i h)}{\delta_i^2}\right] E_j(f_0)$$
$$+6\sum_{i=1}^{p} a_i \sum_{k=0}^{\bar{p}_i} \left(\sum_{j=0}^{p} \frac{a_j w_{i,j,k}}{\delta_i-\delta_k}\left[\frac{1-\exp(-\delta_k h)}{\delta_k} - \frac{1-\exp(-\delta_i h)}{\delta_i}\right]\right) E_k(f_0)$$
$$+12\rho^2 \sum_{i=1}^{p} a_i \sum_{k=0}^{\bar{p}} \left(\sum_{j=0}^{p_i} \frac{e_{i,j}e_{j,k}}{\delta_j-\delta_k}\left[\frac{1-\exp(-\delta_k h)}{(\delta_i-\delta_k)\delta_k} - \frac{1-\exp(-\delta_i h)}{(\delta_i-\delta_k)\delta_i} - \frac{1-\exp(-\delta_j h)}{(\delta_i-\delta_j)\delta_j} + \frac{1-\exp(-\delta_i h)}{(\delta_i-\delta_j)\delta_i}\right]\right) E_k(f_0).$$

After rearranging some terms and making some calculus, one gets:

$$E[Z_h^4 \mid I_0] = m^4h^4 + 6m^2 a_0 h^3 + +6m^2h^2 \sum_{i=1}^{p} \frac{a_i}{\delta_i}(1-\exp(-\delta_i h))E_i(f_0)$$
$$+3a_0^2h^2 + 6a_0 \sum_{i=1}^{p} \frac{a_i}{\delta_i^2}[\exp(-\delta_i h) - 1 + \delta_i h]E_i(f_0)$$
$$+6\sum_{k=0}^{\bar{\bar{p}}} \left[\sum_{i=1}^{p} a_i \left(\sum_{j=0}^{p} \frac{a_j w_{i,j,k}}{\delta_i-\delta_k}\left[\frac{1-\exp(-\delta_k h)}{\delta_k} - \frac{1-\exp(-\delta_i h)}{\delta_i}\right]\right)\right] E_k(f_0)$$
$$+12m\rho h \sum_{j=0}^{\bar{p}} \left(\sum_{i=1}^{p} \frac{a_i e_{i,j}}{\delta_i-\delta_j}\left[\frac{1-\exp(-\delta_j h)}{\delta_j} - \frac{1-\exp(-\delta_i h)}{\delta_i}\right]\right) E_j(f_0)$$
$$+12\rho^2 \sum_{k=0}^{\bar{p}} \left[\sum_{i=1}^{p} a_i \left(\sum_{j=0}^{p_i} \frac{e_{i,j}e_{j,k}}{\delta_j-\delta_k}\left[\frac{1-\exp(-\delta_k h)}{(\delta_i-\delta_k)\delta_k} - \frac{1-\exp(-\delta_i h)}{(\delta_i-\delta_k)\delta_i} - \frac{1-\exp(-\delta_j h)}{(\delta_i-\delta_j)\delta_j} + \frac{1-\exp(-\delta_i h)}{(\delta_i-\delta_j)\delta_i}\right]\right)\right] E_k(f_0).\blacksquare$$