Revisiting Continuous Time Limits of Volatility Processes*

Jihyun Kim and Nour Meddahi†

August 5, 2017

Abstract

In an important contribution, Nelson (1990) studied the convergence of stochastic difference equations like GARCH models to stochastic differential equations as the length of the discrete time intervals between observations goes to zero. In particular, Nelson (1990) proved that the GARCH model converges to a stochastic volatility model. In a different work, Corradi (2000) reconsidered Nelson's work and made some different assumptions and proved that the limiting process of her GARCH process was a deterministic volatility model. The difference among the two papers is in the specification of the GARCH(1,1) parameters as functions of the length of the returns, which are not part of the GARCH model and are ad hoc assumptions. In this note, we argue that instead of making these ad hoc assumptions, one should impose restrictions at the aggregated level, like daily frequencies. More specifically, by imposing the empirical observations that daily data are leptokurtic and that their squares are auto-correlated, we get the same restrictions imposed by Nelson (1990), giving to his results more empirical and theoretical foundations.

Keywords: GARCH, limiting process, GARCH diffusions, aggregate level.

---

*This work was supported by SCOR and Meridiam.

†Toulouse School of Economics, University of Toulouse Capitole, 31000 Toulouse, France

1 Introduction

In a series of seminal papers, Daniel Nelson studied the connection between GARCH models and stochastic volatility models (Nelson (1990, 1996a-b); Nelson and Foster (1994, 1995)). In particular, Nelson (1990) considered the limiting behavior of a GARCH(1,1) process of Bollerslev (1986) when the length of the return vanishes to zero and proved that, under some conditions, the limiting process is a continuous time stochastic volatility model. Nelson (1990) also considered the limiting behavior of EGARCH models (Nelson, 1991) and other ARCH type models (Engle, 1982) and reached the same conclusion, with different stochastic volatility models. Duan (1997) extended Nelson (1990) by considering more general GARCH-type models.

In a different study, Corradi (2000) revisited Nelson (1990) and made some different assumptions and reached a quite different result than Nelson (1990). Indeed, Corradi (2000) proved that the limiting process of her GARCH(1,1) process was a deterministic volatility model.

Both Nelson (1990) and Corradi (2000) are correct, and are based on a GARCH(1,1) model of Bollerslev (1986). The difference of the two papers is in the specification of the GARCH(1,1) parameters as functions of the length of the returns. These functions are not part of the GARCH model of Bollerslev (1986) and are indeed somewhat ad hoc.

One should mention that the economic implications of the two limiting processes are quite different. When the volatility is deterministic, a frictionless market with the underlying asset and a risk free asset is dynamically complete. In contrast, it is not the case when the volatility is stochastic. In the later, one needs to add an asset like a European option to complete the market; see Bajeux and Rochet (1996).

The main goal of this note is to propose some discipline in the specification of the variation of the model's parameters when the length of the returns vanishes. The main idea of the proposal is to look to desirable properties of the model at the macroscopic level. To be more specific, one could consider the daily aggregated returns and assume that some properties of these aggregated return (macroscopic level) are maintained when the length of the high frequency (intra-day) returns (microscopic level) vanishes. For instance, one could impose that some moments of the aggregated returns are roughly constant when the length of the intra-day returns varies. In particular, one could impose that the aggregated returns are leptokurtic as in the data.

ARCH models and their extensions (GARCH, EGARCH, and etc.) as well as stochastic volatility models have been empirically quite successful because they capture well the persistence of volatility, that is the positive auto-correlation of squared returns. Therefore, a possible desirable property to maintain at the macroscopic (daily) level is the positive auto-correlation of the squared daily returns when the length of disaggregated (intra-day) returns vanishes.

In this note, we show that when one wants to maintain the positive auto-correlation of the squared daily returns and assume that the first two unconditional moments do not vary too much when the length of the intra-day returns vanishes, one gets the same assumptions as Nelson (1990) and not those of Corradi (2000). In other words, Nelson's (1990) results have more empirical and theoretical foundations.

2 The Model

Consider the following GARCH(1,1) process introduced by Bollerslev (1986)

$$y_{kh,h} = y_{(k-1)h,h} + r_{kh,h} = y_{(k-1)h,h} + \sigma_{(k-1)h,h}\varepsilon_{kh,h}$$

where $y_{kh,h}$ is a log-price and $r_{kh,h}$ is the log-return over the period $[(k-1)h, kh]$, with

$$\sigma^2_{kh,h} = \omega_h + \alpha_h h^{-1} r^2_{kh,h} + \beta_h \sigma^2_{(k-1)h,h}, \quad \omega_h, \alpha_h, \beta_h > 0$$

$$\varepsilon_{kh,h} \sim \text{i.i.d. } \mathcal{N}(0, h).$$

In the sequel, we will define γ_h as

$$\gamma_h = \alpha_h + \beta_h.$$

Throughout of this note, we assume that the variance process $\{\sigma^2_{kh,h}\}$ is time varying, that is $\gamma_h > 0$, and is stationary with finite mean, that is $\gamma_h < 1$.

2.1 Continuous Time Limits of GARCH Processes in Nelson (1990) and Corradi (2000)

In order to study the continuous time limits of the GARCH(1,1), Nelson (1990) and Corradi (2000) considered the continuous time trajectory of the GARCH(1,1) process via piecewise constant interpolation as

$$y_t^{(h)} = y_{kh,h}, \quad \sigma_t^{2,(h)} = \sigma_{kh,h}^2, \quad kh \le t < (k+1)h,$$

and made additional assumptions on the behavior of the model's parameters $(w_h, \alpha_h, \gamma_h)$ when h varies. Both Nelson (1990) and Corradi (2000) assumed that

$$\omega_h = \omega h + o(h), \quad \gamma_h = 1 - \theta h + o(h) \tag{1}$$

which implies that the conditional mean of variance increment is given by

$$h^{-1} E\left[\sigma_{kh}^{2,(h)} - \sigma_{(k-1)h}^{2,(h)} \middle| y_{(k-1)h}^{(h)}, \sigma_{(k-1)h}^{2,(h)}\right] = \omega - \theta \sigma_{(k-1)h}^{2,(h)} + o_p(h).$$

However, these authors made different assumptions about α_h given that

$$\text{Nelson (1990)}: \quad \alpha_h = \frac{\alpha}{\sqrt{2}}\sqrt{h} + o(\sqrt{h}), \quad \alpha \neq 0, \tag{2}$$

$$\text{Corradi (2000)}: \quad \alpha_h = o(h^\delta), \quad \forall \delta < 1. \tag{3}$$

Each of Eqs. (2) and (3) has a different implication on the conditional variance of variance increment, that is

$$\begin{aligned} h^{-1} E\left[\left(\sigma_{kh}^{2,(h)} - \sigma_{(k-1)h}^{2,(h)}\right)^2 \middle| y_{(k-1)h}^{(h)}, \sigma_{(k-1)h}^{2,(h)}\right] &= 2h^{-1}\alpha_h^2 (\sigma_{(k-1)h}^{2,(h)})^2 + o_p(h) \\ &= \begin{cases} \alpha^2 (\sigma_{(k-1)h}^{2,(h)})^2 + o_p(h), & \text{under Eq. (2)} \\ o_p(h), & \text{under Eq. (3)} \end{cases} \end{aligned}$$

The continuous time limit of the time interpolated GARCH(1,1) process can be obtained from the above two conditional moments with the weak convergence of Markov chains to diffusion processes by Strook and Varadhan (1979, Chapter 11) or by Ethier and Kurtz (1986, Chapter 8). See also Francq and Zakoian (2010, Chapter 12).

In particular, under Eqs. (1) and (2), Nelson proved that the time interpolated GARCH(1,1) process $(y_t^{(h)}, \sigma_t^{2,(h)})$ converges in distribution to the stochastic volatility model (y_t, σ_t^2) given by

$$dy_t = \sigma_t dW_{1,t}, \quad d\sigma_t^2 = (\omega - \theta\sigma_t^2)dt + \alpha\sigma_t^2 dW_{2,t}$$

where $W_{1,t}$ and $W_{2,t}$ are two independent standard Brownian processes.

In contrast, Corradi (2000) proved that under Eqs. (1) and (3), the time interpolated GARCH(1,1) process $(y_t^{(h)}, \sigma_t^{2,(h)})$ converges in distribution to the following model

$$dy_t = \sigma_t dW_{1t}, \quad d\sigma_t^2 = (\omega - \theta\sigma_t^2)dt,$$

where, in particular, the limiting variance process $\{\sigma_t^2\}$ is deterministic.

2.2 Aggregated Properties of the GARCH(1,1) Process

Without loss of generality, we will refer to $n = 1, 2, ...$ as days while one observes intra-day returns of length h. The log-return of the day n with intra-day returns of length h is denoted by $R_{n,h}$ and defined by

$$R_{n,h} \equiv y_{n,h} - y_{n-1,h} = \sum_{k=1}^{1/h} (y_{n-1+kh,h} - y_{n-1+(k-1)h,h}) = \sum_{k=1}^{1/h} r_{n-1+kh,h}.$$

Before characterizing macroscopic properties of the process $R_{n,h}$, it is worth to recap some results about the process $r_{kh,h}$. The process $r_{kh,h}$ is a GARCH(1,1) of Bollerslev (1986) who proved the following properties when the fourth moment of the returns is assumed to be finite:

$$E[r_{kh,h}] = 0, \quad E[r^2_{kh,h}] = hE[\sigma^2_{kh,h}] = \frac{h\omega_h}{1-\gamma_h}, \quad E[r^3_{kh,h}] = 0, \tag{4}$$

$$E[r^4_{kh,h}] = 3\frac{h^2\omega_h^2(1+\gamma_h)}{(1-\gamma_h)(1-\gamma_h^2-2\alpha_h^2)}, \quad Var[r^2_{kh,h}] = 2\frac{h^2\omega_h^2(1-\gamma_h^2+\alpha_h^2)}{(1-\gamma_h)^2(1-\gamma_h^2-2\alpha_h^2)},$$

$$\begin{aligned}
&Cov[r_{kh,h}, r_{lh,h}] = 0 \quad \forall k \neq l, \\
&Cov[r^2_{kh,h}, r^2_{(k+1+j)h,h}] = \gamma_h^j Cov[r^2_{kh,h}, r^2_{(k+1)h,h}], \qquad (5) \\
&Cov[r^2_{kh,h}, r^2_{(k+1)h,h}] = \frac{\alpha_h(1+\alpha_h\gamma_h-\gamma_h^2)}{1-\gamma_h^2+\alpha_h^2} Var[r^2_{kh,h}], \\
&Cov[r^2_{kh,h}, r_{lh,h}] = 0 \quad \forall k, l.
\end{aligned}$$

We will now characterize macroscopic properties about the process $R_{n,h}$. Given that the difference between the specifications of Nelson (1990) and Corradi (2000) are related to the conditional variance $\sigma^2_{kh,h}$ process, these properties will be written in terms of the moments of $\sigma^2_{kh,h}$.

Proposition 1 *One has*

$$E[R_{n,h}] = 0, \tag{6}$$

$$Var[R_{n,h}] = E[R^2_{n,h}] = E[\sigma^2_{kh,h}], \tag{7}$$

$$\frac{E[R^4_{n,h}]}{(E[R^2_{n,h}])^2} = 3P(\gamma_h, h)\frac{Var[\sigma^2_{kh,h}]}{(E[\sigma^2_{kh,h}])^2} + 3, \tag{8}$$

$$Cov[R^2_{n,h}, R^2_{n+1,h}] = Q(\gamma_h, h)Var[\sigma^2_{h,h}], \tag{9}$$

$$Cov[R^2_{n,h}, R^2_{n+j,h}] = (\gamma_h^{1/h})^{j-1} Cov[R^2_{n,h}, R^2_{n+1,h}], \tag{10}$$

where

$$P(\gamma_h, h) = h + 2h\frac{\gamma_h^2}{(1-\gamma_h)^2}(h\gamma_h^{1/h} - h + 1 - \gamma_h),$$
$$Q(\gamma_h, h) = \frac{(1-\gamma_h^{1/h})^2}{(1-\gamma_h)^2}\ h^2\gamma_h.$$

Let us now look to the desirable properties of the aggregated process $\{R_{n,h}\}$ when $h \to 0$.

Property P1: The first two moments of $\{R_{n,h}\}$ have limits when $h \to 0$, with the limit of the variance being non-zero.

Property P2: $Cov[R^2_{n,h}, R^2_{n+j,h}]$ has non-zero limit when $h \to 0$ for any finite j.

Property P3: $Cov[R^2_{n,h}, R^2_{n+j,h}]$ has a zero limit when $h \to 0$ and $j \to +\infty$.

Property P4: $\{R_{n,h}\}$ is leptokurtic when $h \to 0$.

Let us start with Property P1. The mean of the aggregated return is zero. The variance of the aggregated return is given by Eq. (7). The AR(1) specification of the variance process $\sigma^2_{kh,k}$ implies that

$$E[R^2_{n,h}] = \frac{\omega_h}{1-\gamma_h}.$$

Property P1 implies that there exists a non-zero constant V such that when h is small, one has

$$\frac{\omega_h}{1-\gamma_h} = V + o(1), \quad 0 < V < \infty. \tag{11}$$

Let us now focus on the behavior of γ_h. For $j > 1$, Eq. (10) implies that the covariance between $R^2_{n,h}$ and $R^2_{n+j,h}$ is driving by $\gamma_h^{1/h}$ when h varies. We have assumed previously that the variance process $\{\sigma^2_{kh,h}\}$ is a nontrivial stationary process with finite mean, and therefore, $0 < \gamma_h < 1$ for all $h > 0$. Hence, one has $0 < \gamma_h^{1/h} < 1$. Consequently, under the assumption that $\gamma_h^{1/h}$ has a limit when $h \to 0$, one gets

$$0 \leq \lim_{h\to 0}\gamma_h^{1/h} \leq 1.$$

The second desirable property, P2, is to keep serial correlation in the squared returns. Property P2, jointly with Eq. (10), implies that $\gamma_h^{1/h}$ has a non-zero limit when h goes to zero in order to have non-zero serial correlation between, for instance, $R^2_{n+2,h}$ and $R^2_{n,h}$.

Likewise, a desirable property is that the correlation between $R^2_{n+j,h}$ and $R^2_{n,h}$ will vanish when $j \to +\infty$. Therefore, Property P3 implies $\lim_{h\to 0}\gamma_h^{1/h} < 1$.

Consequently, by combining the properties P2 and P3, one gets

$$0 < \lim_{h \to 0} \gamma_h^{1/h} < 1,$$

and hence, one needs

$$\gamma_h^{1/h} = C + o(1), \quad 0 < C < 1. \tag{12}$$

Lemma 1 *Eq. (12) holds if and only if*

$$\gamma_h = 1 - \theta h + o(h), \quad \text{with} \ \ \theta = -\log(C) > 0. \tag{13}$$

It follows from Lemma 1 with Eq. (10) that the properties P2 and P3 hold if and only if Eq. (13) holds as long as $Cov[R_{n,h}^2, R_{n+1,h}^2]$ has non-zero limit, which will be shown in Lemma 4 below.

We are now able to characterize the behavior of ω_h.

Lemma 2 *Given Eq. (13), Eq. (11) holds if and only if*

$$\omega_h = \omega h + o(h), \quad \text{with} \ \ \omega = V\theta. \tag{14}$$

Lemma 2 implies that given Eq. (13) the variance of $R_{n,h}$ has a time invariant limit if and only if Eq. (14) holds. Combining this with Lemma 1 and $E[R_{n,h}] = 0$, we may conclude that the properties P1, P2, and P3 hold if and only if Eqs. (13) and (14) hold as long as $Cov[R_{n,h}^2, R_{n+1,h}^2]$ has non-zero limit.

We will now study Property P4 as well as $Cov[R_{n,h}^2, R_{n+1,h}^2]$. We first analyze the behaviors of $P(\gamma_h, h)$ in Eq. (8) as well as the behavior of $Q(\gamma_h, h)$ that appears in Eq. (9) when h is small and Eq (13) is given.

Lemma 3 *Given Eq. (13), one has*

$$P(\gamma_h, h) = \frac{2(\exp(-\theta) - 1 + \theta)}{\theta^2} + o(1), \quad Q(\gamma_h, h) = \frac{(1 - \exp(-\theta))^2}{\theta^2} + o(1)$$

with

$$\frac{2(\exp(-\theta) - 1 + \theta)}{\theta^2} > 0, \quad \frac{(1 - \exp(-\theta))^2}{\theta^2} > 0$$

for any $\theta > 0$.

This lemma implies that the behavior of $Cov[R^2_{n,h}, R^2_{n+1,h}]$ and the excess kurtosis depend on $Var[\sigma^2_{h,h}]$ and θ. In particular, P2 holds if and only if P4 holds due to Proposition 1 and Lemmas 1-3. Moreover, $Var[\sigma^2_{h,h}]$ should have a non-zero limit to satisfy both P2 and P4. Therefore, one may assume

$$Var[\sigma^2_{h,h}] = \sigma^2 + o(1).$$

On the other hand, it is well known that

$$Var[\sigma^2_{h,h}] = \frac{2\alpha_h^2}{1-\gamma_h^2-2\alpha_h^2}\left(\frac{w_h}{1-\alpha_h-\beta_h}\right)^2,$$

provided the denomitor is positive, that is the fourth moment of the high frequency returns is bounded.

Lemma 4 *Given Eqs. (13) and (14), one has*

$$Var[\sigma^2_{h,h}] = \left[\frac{\theta h}{\alpha_h^2} - 1 + o\left(\frac{\theta h}{\alpha_h^2}\right)\right]^{-1}(V^2 + o(1)). \tag{15}$$

Consequently, $Var[\sigma^2_{h,h}]$ *has a non-zero limit if and only if*

$$\alpha_h^2 = Dh + o(h), \quad \text{with} \quad D > 0, \tag{16}$$

that is

$$\alpha_h = \sqrt{D}\sqrt{h} + o(\sqrt{h}).$$

Lemma 4, jointly with Lemmas 1-3, implies that the desirable properties of keeping heteroskedasticity, P2, or excess kurtosis, P4, at the aggregated level hold if and only if the assumption made by Nelson (1990) is satisfied. On the other hand, Eq. (15) in Lemma 4 implies that $Var[\sigma^2_{h,h}]$ has zero limit under assumption Eq. (3) made by Corradi (2000). More importantly, neither P2 nor P4 holds under the Corradi's assumption.

Proposition 2 *The properties, P1-P4, hold if and only if the assumption made by Nelson (1990) hold. On the other hand, the assumption made by Corradi (2000) implies P1 and P3 but neither P2 nor P4.*

3 Conclusion

In this note, we proved that when one studies the limiting behavior of a GARCH process when the length of intra-day returns goes to zero, one should impose restrictions at the aggregated level like daily frequencies, where the restrictions are empirically motivated. By doing so, we get the Nelson's (1990) results which are more realistic than those of Corradi (2000).

Appendix

Proof of Proposition 1. The process $\{r_{kh,h}\}$ is a GARCH(1,1) of Bollerslev (1986) who proved the first equality in Eq. (4), from which one has Eq. (6). Likewise,

$$Var[R_{n,h}] = \sum_{k=1}^{1/h} Var[r_{kh,h}] + 2 \sum_{1\leq k<l\leq 1/h} Cov[r_{kh,h}, r_{lh,h}] = \frac{Var[r_{kh,h}]}{h} = E[\sigma^2_{kh}],$$

that is Eq. (7). On the other hand, one has

$$\begin{aligned}
E[R^4_{n,h}] &= \sum_{k=1}^{1/h} E[r^4_{kh,h}] + 6 \sum_{1\leq k<l\leq 1/h} E[r^2_{kh,h} r^2_{lh,h}] \\
&= \frac{E[r^4_{kh,h}]}{h} + 6 \left[\sum_{k=2}^{1/h} (1-k+1/h) Cov[r^2_{h,h}, r^2_{kh,h}]\right] + 6(E[r^2_{kh,h}])^2 \sum_{k=2}^{1/h}(1-k+1/h) \\
&= \frac{E[r^4_{kh,h}]}{h} + 6Cov[r^2_{h,h}, r^2_{2h,h}] \left[\sum_{k=2}^{1/h}(1-k+1/h)\gamma_h^{k-1}\right] + 3(E[r^2_{kh,h}])^2 \frac{(1-h)}{h^2},
\end{aligned}$$

where the last equality is implyed by Eq. (5).

Observe that $E[r^4_{kh,h}] = 3h^2 E[\sigma^4_{kh,h}]$ while $Cov[r^2_{h,h}, r^2_{2h,h}] = h^2 Cov[\sigma^2_{h,h}, \sigma^2_{2h,h}]$. In addition, one can prove easily the following formula:

$$\sum_{k=2}^{1/h}(1-k+1/h)\gamma_h^{k-1} = \frac{\gamma_h}{h(1-\gamma_h)^2}(1-h-\gamma_h+h\gamma_h^{1/h}) \equiv A(\gamma_h, h).$$

Consequently,

$$\frac{E[R^4_{n,h}]}{(E[R^2_{n,h}])^2} = 3h\frac{E[\sigma^4_{kh,h}]}{(E[\sigma^2_{kh,h}])^2} + 6h^2 \frac{Cov[\sigma^2_{h,h}, \sigma^2_{2h,h}]}{(E[\sigma^2_{kh,h}])^2} A(\gamma_h, h) + 3(1-h).$$

The process $\sigma^2_{kh,k}$ is an autoregressive process of order one with

$$\sigma^2_{kh,h} = \omega_h + \alpha_h \sigma^2_{(k-1)h,h} \frac{\varepsilon^2_{(k-1)h,h}}{h} + \beta_h \sigma^2_{(k-1)h,h} = \omega_h + \gamma_h \sigma^2_{(k-1)h,h} + \alpha_h \sigma^2_{(k-1)h,h}\left(\frac{\varepsilon^2_{kh,h}}{h} - 1\right).$$

Therefore,

$$Cov[\sigma^2_{kh,h}, \sigma^2_{(k+1)h,h}] = \gamma_h Var[\sigma^2_{kh,h}].$$

Likewise, $Var[\sigma^2_{kh,h}] = E[\sigma^4_{kh,h}] - (E[\sigma^2_{kh,h}])^2$, which implies that

$$\begin{aligned}
\frac{E[R^4_{n,h}]}{(E[R^2_{n,h}])^2} &= 3h\frac{Var[\sigma^2_{kh,h}] + (E[\sigma^2_{kh,h}])^2}{(E[\sigma^2_{kh,h}])^2} + 6h^2 \frac{\gamma_h Var[\sigma^2_{h,h}]}{(E[\sigma^2_{kh,h}])^2} A(\gamma_h, h) + 3(1-h) \\
&= 3h\frac{Var[\sigma^2_{kh,h}]}{(E[\sigma^2_{kh,h}])^2} + 6h^2 \frac{\gamma_h Var[\sigma^2_{h,h}]}{(E[\sigma^2_{kh,h}])^2} A(\gamma_h, h) + 3 \\
&= 3h\frac{Var[\sigma^2_{kh,h}]}{(E[\sigma^2_{kh,h}])^2} [1 + 2h\gamma_h A(\gamma_h, h)] + 3,
\end{aligned}$$

that is Eq. (8).

The absence of leverage effect implies that

$$Cov\left[R_{n,h}^2, R_{n+1,h}^2\right] = Cov\left[\sum_{k=1}^{1/h} r_{kh,h}^2, \sum_{l=1}^{1/h} r_{lh+1,h}^2\right] = Cov[r_{kh,h}^2, r_{(k+1)h,h}^2]\left(\sum_{k=1}^{1/h}\sum_{l=1}^{1/h}\gamma_h^{1/h+l-k-1}\right).$$

One can show that

$$\left(\sum_{k=1}^{1/h}\sum_{l=1}^{1/h}\gamma_h^{1/h+l-k-1}\right) = \frac{(1-\gamma_h^{1/h})^2}{(1-\gamma_h)^2}.$$

Hence,

$$Cov\left[R_{n,h}^2, R_{n+1,h}^2\right] = \frac{(1-\gamma_h^{1/h})^2}{(1-\gamma_h)^2} h^2 Cov[\sigma_{kh,h}^2, \sigma_{(k+1)h,h}^2] = \frac{(1-\gamma_h^{1/h})^2}{(1-\gamma_h)^2} h^2 \gamma_h Var[\sigma_{kh,h}^2],$$

that is Eq. (9).

Finally, we know that the aggregation of a GARCH(1,1) with finite fourth moment leads to an ARMA(1,1) representation of the squared returns prcess (Drost and Nijman (1993), Drost and Werker (1966), Meddahi and Renault (2004)), which implies that

$$Cov[R_{n,h}^2, R_{n+j,h}^2] = \theta(h)^{(j-1)} Cov[R_{n,h}^2, R_{n+1,h}^2],$$

where $\theta(h)$ is the autoregressive parameter of the ARMA(1,1) representation of $R_{n,h}^2$. It turns out that Drost and Nijman (1993) proved that $\theta(h)$ equals the high frequency autoregressive parameter (here γ_h) power the number of aggregated periods, here $1/h$, that is one has $\theta(h) = \gamma_h^{1/h}$. Therefore, one gets Eq. (10).

Proof of Lemma 1. Eq. (12) implies

$$\log(\gamma_h) = h\log(C + o(1)) = h(\log(C) + o(1)) = -h\theta + o(h),$$

and hence,

$$\gamma_h = \exp(-h\theta + o(h)) = 1 - h\theta + o(h).$$

The converse is trivial by the definition of the exponential function.

Proof of Lemma 2. Eqs. (11) and (13) imply

$$\omega_h = (V + o(h))(\theta h + o(h)) = V\theta h + o(h).$$

The converse is trivial.

Proof of Lemma 3. When $h \to 0$ and under Eq. (13), one has

$$\gamma_h^{1/h} = \exp(-\theta) + o(1), \quad (1-\gamma_h)^2 = \theta^2 h^2 (1 + o(1)).$$

Therefore,

$$Q(\gamma_h, h) = \frac{(1-\exp(-\theta)+o(1))^2 h^2 (1-\theta h + o(h))}{\theta^2 h^2 (1+o(1))} = \frac{(1-\exp(-\theta))^2}{\theta^2} + o(1).$$

Likewise,

$$\begin{aligned}\frac{P(\gamma_h, h)}{h} &= 1 + 2\frac{(1-2\theta h + o(h))}{(\theta h + o(h))^2}(h(\exp(-\theta)+o(1)) - h + 1 - 1 + \theta h + o(h)) \\ &= 1 + 2\frac{(1-2\theta h + o(h))}{\theta^2 h (1+o(1))}((\exp(-\theta) - 1 + \theta + o(1)).\end{aligned}$$

Hence,

$$\begin{aligned}P(\gamma_h, h) &= h + 2\frac{(1-2\theta h + o(h))}{\theta^2(1+o(1))}((\exp(-\theta) - 1 + \theta + o(1)) \\ &= h + 2\frac{(\exp(-\theta) - 1 + \theta)}{\theta^2} + o(1) \\ &= 2\frac{(\exp(-\theta) - 1 + \theta)}{\theta^2} + o(1).\end{aligned}$$

Proof of Lemma 4. When $h \to 0$ and under Eqs. (13) and (14), one has

$$\begin{aligned}Var[\sigma_{h,h}^2] &= \frac{2\alpha_h^2}{2\theta h - 2\alpha_h^2 + o(h)}(V + o(1))^2 \\ &= \frac{1}{\theta h/\alpha_h^2 - 1 + o(\theta h/\alpha_h^2))}(V^2 + o(1)),\end{aligned}$$

that is Eq. (15).

$Var[\sigma_{h,h}^2]$ has a limit if $\theta h/\alpha_h^2 - 1$ has a limit. Non-negativity of $Var[\sigma_{h,h}^2]$ implies that this limit should be non-negative, implying that $\theta h/\alpha_h^2 \geq 1$ for small h. At the limit, $Var[\sigma_{h,h}^2]$ should have a non-zero limit, which excludes an infinite limit for $\theta h/\alpha_h^2$. In other words, the limit of $\theta h/\alpha_h^2$ should be finite and strictly larger than one. Therefore, one needs to have Eq. (16) (with $D < \theta$). The converse is trivial.

Proof of Proposition 2. It follows immediately from Lemmas 1-4 with Proposition 1.

References

Bajeux-Besnainou, I. and J. C. Rochet (1996), "Dynamic Spanning: Are Options and Appropriate Instrument?" *Mathematical Finance*, 6, 1-17.

Bollerslev, T. (1986), "Generalized Autoregressive Conditional Heteroskedasticity," *Journal of Econometrics*, 31, 307-327.

Corradi, V. (2000), "Reconsidering the Continuous Time Limit of the GARCH(1,1) Process," *Journal of Econometrics*, 96, 145-153.

Drost, F.C. and T.E. Nijman (1993), "Temporal Aggregation of GARCH Processes," *Econometrica*, 61, 909-927.

Drost, F.C. and J.C. Werker (1996), "Closing the GARCH Gap: Continuous Time GARCH Modeling," *Journal of Econometrics*, 74, 31-58.

Duan, J.C. (1997), "Augmented GARCH(p,q) Process and its Diffusion Limit," *Journal of Econometrics*, 79, 97-127.

Engle, R.F. (1982), "Autoregressive Conditional Heteroskedasticity with Estimates of the Variance of United Kingdom Inflation," *Econometrica*, 50, 987-1007.

Ethier, S.N. and T.G. Kurtz (1986), *Markov Processes: Characterization and Convergence*, Wiley, New York

Francq, C. and J.-M. Zakoian (2010), *GARCH Models: Structure, Statistical Inference and Financial Applications*, Wiley.

Meddahi, N. and E. Renault (2004), "Temporal Aggregation of Volatility Models," *Journal of Econometrics*, 119, 355-379.

Nelson, D.B. (1990), "ARCH Models as Diffusion Approximations," *Journal of Econometrics*, 45, 7-39.

Nelson, D.B. (1991), "Conditional Heteroskedasticity in Asset Returns: A New Approach," *Econometrica*, 59, 347-370.

Nelson, D.B. (1996-a), "Asymptotically Optimal Smoothing with ARCH Models," *Econometrica*, 64, 561-573.

Nelson, D.B. (1996-b), "Asymptotic Filtering Theory for Multivariate ARCH Models," *Journal of Econometrics*, 71, 1-47.

Nelson, D.B. and D.P. Foster (1994), "Asymptotic Filtering Theory for Univariate ARCH models," *Econometrica*, 62, 1-41.

Nelson, D.B. and D.P. Foster (1995), "Filtering and Forecasting with Misspecified ARCH Models II: Making the Right Forecast with the Wrong Model," *Journal of Econometrics*, 67, 303-335.

Strook, D.W. and S.R.S. Varadhan (1979), *Multidimensional Diffusion Processes*, Springer, Berlin.