Bootstrapping high-frequency jump tests: Supplementary Appendix*

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This supplementary appendix is organized as follows. In Section S1, we first provide an auxiliary lemma and then provide proofs of the general bootstrap results appearing in Section 3 of the main paper. In Section S2, we establish the results appearing in Section 4 of the main paper. In particular, this section contains the asymptotic expansion of the cumulants of the asymptotic test statistic T_n and its bootstrap versions T_n^* and $\bar{T}_n^*$. The limits of these cumulants are derived by relying on some auxiliary lemmas that are introduced and proved in this section of the appendix. Detailed formulas useful for the implementation of the log version of our tests are provided in Appendix S3. Finally, Section S4 presents the theoretical justification for the local Gaussian bootstrap when applied to two alternative jump tests: the test of Podolskij and Ziggel (2010) and the big jumps test of Lee and Hannig (2010).

Appendix S1: Proofs of results in Section 3

We first derive the first and second order bootstrap moments of $(RV_n^*, BV_n^*)'$. Note that since $r_i^* = \sqrt{\hat{v}_i^n} \cdot \eta_i$, we can write

$$RV_n^* = \sum_{i=1}^{n} \hat{v}_i^n \cdot u_i \quad \text{and} \quad BV_n^* = \frac{1}{k_1^2} \sum_{i=2}^{n} \left(\hat{v}_{i-1}^n\right)^{1/2} \left(\hat{v}_i^n\right)^{1/2} \cdot w_i$$

where $u_i \equiv \eta_i^2$ and $w_i \equiv |\eta_{i-1}| \, |\eta_i|$, with $\eta_i \sim$ i.i.d. $N(0,1)$. The bootstrap moments of $(RV_n^*, BV_n^*)'$ depend on the moments and dependence properties of (u_i, w_i). The proof is trivial and is omitted for brevity.

Lemma S1.1 *If $r_i^* = \sqrt{\hat{v}_i^n} \cdot \eta_i$, $\quad i = 1, \ldots, n$, where $\eta_i \sim$ i.i.d. $N(0,1)$, then*

(a1) $E^*(RV_n^*) = \sum_{i=1}^{n} \hat{v}_i^n$.

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(a2) $E^*\left(BV_n^*\right) = \sum\limits_{i=2}^{n} \left(\hat{v}_{i-1}^n\right)^{1/2} \left(\hat{v}_i^n\right)^{1/2}.$

(a3) $Var^*\left(\sqrt{n}RV_n^*\right) = 2n \sum\limits_{i=1}^{n} \left(\hat{v}_i^n\right)^2.$

(a4) $Var^*\left(\sqrt{n}BV_n^*\right) = \left(k_1^{-4}-1\right) n \sum\limits_{i=2}^{n} \left(\hat{v}_i^n\right)\left(\hat{v}_{i-1}^n\right) + 2\left(k_1^{-2}-1\right) n \sum\limits_{i=3}^{n} \left(\hat{v}_i^n\right)^{1/2} \left(\hat{v}_{i-1}^n\right) \left(\hat{v}_{i-2}^n\right)^{1/2}.$

(a5) $Cov^*\left(\sqrt{n}RV_n^*, \sqrt{n}BV_n^*\right) = n \sum\limits_{i=2}^{n} \left(\hat{v}_i^n\right)^{3/2} \left(\hat{v}_{i-1}^n\right)^{1/2} + n \sum\limits_{i=2}^{n} \left(\hat{v}_i^n\right)^{1/2} \left(\hat{v}_{i-1}^n\right)^{3/2}.$

Proof of Theorem 3.1. We first show that

$$Z_n^* \equiv \Sigma_n^{*^{-1/2}} \sqrt{n} \begin{pmatrix} RV_n^* - E^*\left(RV_n^*\right) \\ BV_n^* - E^*\left(BV_n^*\right) \end{pmatrix} \xrightarrow{d^*} N\left(0, I_2\right), \tag{S1.1}$$

in prob-P. Write

$$Z_n^* = \Sigma_n^{*^{-1/2}} \sqrt{n} \sum_{i=1}^{n} D_i e_i^* = \sqrt{n} \sum_{i=1}^{n} z_i^*,$$

with $z_i^* = \Sigma_n^{*-1/2} D_i e_i^*$, and

$$D_i = \begin{pmatrix} \hat{v}_i^n & 0 \\ 0 & \frac{1}{k_1^2}\left(\hat{v}_i^n\right)^{1/2}\left(\hat{v}_{i-1}^n\right)^{1/2} \end{pmatrix}, \quad \text{and} \quad e_i^* = \begin{pmatrix} u_i - E^*(u_i) \\ w_i - E^*(w_i) \end{pmatrix},$$

where we set $\hat{v}_0^n = 0$ and where $u_i = \eta_i^2$ and $w_i = |\eta_i|\,|\eta_{i-1}|$ and $\eta_i \sim$ i.i.d. $N(0,1)$. Note that e_i^* is a zero mean vector that is lag-1-dependent. We follow Pauly (2011) and rely on a modified Cramer-Wold device to establish the bootstrap CLT. Let $D = \{\lambda_k : k \in \mathbb{N}\}$ be a countable dense subset of the unit circle of $\mathbb{R}^2$. We have to show that for any $\lambda \in D$, $\lambda' Z_n^* \xrightarrow{d^*} N(0,1)$, in prob-$P$, as $n \to \infty$.

From Lemma 3.1, we have $Var^*(\lambda' Z_n^*) = 1$ for all n. Hence, to conclude, it remains to establish that $\lambda' Z_n^*$ is asymptotically normally distributed, conditionally on the original sample and with probability P approaching one. Since z_i^*'s are lag-1-dependent, we adopt the large-block-small-block type of argument to prove this central limit result (see Shao (2010) for an example of this idea). The large blocks are made of L_n successive observations followed by a small block that is made of a single element.

Let $\ell_n = \left[\frac{n}{L_n+1}\right]$. Define the (large) blocks $\mathcal{L}_j = \{i \in \mathbb{N} : (j-1)(L_n+1)+1 \le i \le j\,(L_n+1)-1\}$, where $1 \le j \le \ell_n$ and $\mathcal{L}_{\ell_n+1} = \{i \in N : \ell_n(L_n+1)+1 \le i \le n\}$. Let $U_j^* = \sum_{i \in \mathcal{L}_j} \lambda' z_i^*$, $j = 1, \ldots, \ell_n+1$. Clearly,

$$\lambda' Z_n^* = \sqrt{n} \sum_{j=1}^{\ell_n+1} U_j^* + \sqrt{n} \sum_{j=1}^{\ell_n} \lambda' z_{j(L_n+1)}^*.$$

Next, we show that under Condition A,

(i) $\sqrt{n} \sum_{j=1}^{\ell_n} \lambda' z_{j(L_n+1)}^* = o_{P^*}(1)$, in prob-$P$; and

(ii) for some $\delta > 0$,

$$\sum_{j=1}^{\ell_n+1} E^* \left|\sqrt{n} U_j^*\right|^{2+\delta} \xrightarrow{P} 0.$$

This latter is sufficient to deduce that $\sqrt{n} \sum_{j=1}^{\ell_n+1} U_j^* \xrightarrow{d^*} N(0,1)$, in prob-$P$, since $\{U_j\}$ form an independent array, conditionally on the sample. The expected result then follows from (i). Let us establish (i). Since $E^*(z_i^*) = 0$ for all i, it suffices to show that $Var^*\left(\sqrt{n}\sum_{j=1}^{\ell_n}\lambda' z^*_{j(L_n+1)}\right) = o_P(1)$. Letting $\Omega_n^* \equiv Var^*\left(\sqrt{n}\sum_{j=1}^{\ell_n} D_{j(L_{n+1})} e^*_{j(L_n+1)}\right)$, by the Cauchy-Schwarz inequality, we have:

$$\left\| Var^*\left(\sqrt{n}\sum_{j=1}^{\ell_n} z^*_{j(L_n+1)}\right)\right\| = \left\|\lambda' {\Sigma_n^*}^{-1/2}\Omega_n^* {\Sigma_n^*}^{-1/2}\lambda\right\| \le \left\|\Sigma_n^{*-1/2}\right\|^2 \|\Omega_n^*\|.$$

Condition A and Lemma 3.1 ensure that $\Sigma_n^* \xrightarrow{P} \begin{pmatrix} 2 & 2 \\ 2 & \theta \end{pmatrix} IQ$ which is positive definite almost surely. Hence ${\Sigma_n^*}^{-1/2} = O_P(1)$. Turning to Ω_n^*, since $L_n \ge 1$ for n large enough, $z^*_{j(L_n+1)}$'s are independent along j conditionally on the sample so that

$$\Omega_n^* = n\sum_{j=1}^{\ell_n} D_{j(L_{n+1})} E^*\left(e^*_{j(L_n+1)} e^{*\prime}_{j(L_n+1)}\right) D'_{j(L_{n+1})}.$$

By the triangle and the Cauchy-Schwarz inequalities, we have:

$$\|\Omega_n^*\| \le n\sum_{j=1}^{\ell_n}\left\|D_{j(L_{n+1})}\right\|^2 \left\|E^*\left(e^*_{j(L_n+1)} e^{*\prime}_{j(L_n+1)}\right)\right\| \le Cn\sum_{j=1}^{\ell_n}\left\|D_{j(L_{n+1})}\right\|^2,$$

where C is a generic constant. Hence,

$$\begin{aligned}
\|\Omega_n^*\| &\le Cn\sum_{j=1}^{\ell_n}\left(\left(\hat{v}^n_{j(L_n+1)}\right)^2 + \tfrac{1}{k_1^4}\left(\hat{v}^n_{j(L_n+1)}\right)\left(\hat{v}^n_{j(L_n+1)-1}\right)\right) \\
&\le Cn\sum_{j=1}^{\ell_n}\left(\hat{v}^n_{j(L_n+1)}\right)^2 + C\left(n\sum_{j=1}^{\ell_n}\left(\hat{v}^n_{j(L_n+1)}\right)^2\right)^{1/2}\left(n\sum_{i=1}^{n}\left(\hat{v}^n_i\right)^2\right)^{1/2} \\
&= o_P(1) + o_P(1)O_P(1) = o_P(1)
\end{aligned}$$

with the equalities following from Condition A. Next, we verify (ii). Let $\delta > 0$. For any $1 \le j \le \ell_n + 1$, we have

$$\left|U_j^*\right|^{2+\delta} = \left|\sum_{i\in\mathcal{L}_j}\lambda' z_i^*\right|^{2+\delta} \le L_n^{1+\delta}\left\|\Sigma_n^{*-1/2}\right\|^{2+\delta}\sum_{i\in\mathcal{L}_j}\|D_i\|^{2+\delta}\|e_i^*\|^{2+\delta}$$

where the inequality follows from the Jensen's and the Cauchy-Schwarz inequalities. It follows that

$$E^*\left|U_j^*\right|^{2+\delta} \le L_n^{1+\delta}\left\|\Sigma_n^{*-1/2}\right\|^{2+\delta}\sum_{i\in\mathcal{L}_j}\|D_i\|^{2+\delta}E^*\|e_i^*\|^{2+\delta} \le CL_n^{1+\delta}\left\|\Sigma_n^{*-1/2}\right\|^{2+\delta}\sum_{i\in\mathcal{L}_j}\|D_i\|^{2+\delta},$$

implying that

$$\begin{aligned}
\sum_{j=1}^{\ell_n+1} E^*\left|\sqrt{n}U_j^*\right|^{2+\delta} &\le Cn^{1+\delta/2}L_n^{1+\delta}\left\|\Sigma_n^{*-1/2}\right\|^{2+\delta}\sum_{j=1}^{\ell_n+1}\sum_{i\in\mathcal{L}_j}\|D_i\|^{2+\delta} \\
&\le Cn^{1+\delta/2}L_n^{1+\delta}\left\|\Sigma_n^{*-1/2}\right\|^{2+\delta}\sum_{j=1}^{\ell_n+1}\sum_{i\in\mathcal{L}_j}\left(\left(\hat{v}_i^n\right)^{(2+\delta)} + \left(\hat{v}_i^n\right)^{\frac{2+\delta}{2}}\left(\hat{v}_{i-1}^n\right)^{\frac{2+\delta}{2}}\right) \\
&\le Cn^{1+\delta/2}L_n^{1+\delta}\left\|\Sigma_n^{*-1/2}\right\|^{2+\delta}\sum_{i=1}^{n}\left(\left(\hat{v}_i^n\right)^{(2+\delta)} + \left(\hat{v}_i^n\right)^{\frac{2+\delta}{2}}\left(\hat{v}_{i-1}^n\right)^{\frac{2+\delta}{2}}\right) \\
&\le C\left\|\Sigma_n^{*-1/2}\right\|^{2+\delta} n^{\alpha(1+\delta)-\delta/2}\left(n^{1+\delta}\sum_{i=1}^{n}\left(\hat{v}_i^n\right)^{(2+\delta)}\right) = O_P\left(n^{\alpha(1+\delta)-\delta/2}\right),
\end{aligned}$$

where the second inequality follows from the Jensen's inequality (recall that C is generic constant) and the last one follows from the Cauchy-Schwarz inequality, given that $L_n = Cn^{\alpha}$. Since $\alpha \in [0, \frac{3}{7})$, $\frac{2\alpha}{1-2\alpha} \in [0, 6)$. Choosing any $\delta \in \left(\frac{2\alpha}{1-2\alpha}, 6\right)$ ensures the last equality, given Condition A(i)) and (ii). This establishes (S1.1).

By the delta method, we can claim that $\sqrt{n}(RV_n^* - BV_n^* - E^*(RV_n^* - BV_n^*))/\sqrt{V_n^*} \xrightarrow{d^*} N(0,1)$ in prob-P, with $V_n^* = Var^*(\sqrt{n}(RV_n^* - BV_n^*))$. Therefore, to conclude, it suffices to show that $\hat{V}_n^* - V_n^* = o_{P^*}(1)$, in prob-$P$. From Lemma 3.1 and Condition A(i), $V_n^* \xrightarrow{P} \tau IQ$. Hence, it suffices to show that $\widehat{IQ}_n^* = IQ + o_{P^*}(1)$, in prob-$P$. We can claim this by observing that $E^*\left(\widehat{IQ}_n^*\right) = IQ + o_P(1)$ and $Var^*\left(\widehat{IQ}_n^*\right) = o_P(1)$. Indeed, it is not hard to obtain that $E^*\left(\widehat{IQ}_n^*\right) = n\sum_{i=3}^{n}(\hat{v}_i^n)^{2/3}(\hat{v}_{i-1}^n)^{2/3}(\hat{v}_{i-2}^n)^{2/3}$ and that

$$
\begin{aligned}
Var^*\left(\widehat{IQ}_n^*\right) &= C\Bigg(n^2\sum_{i=3}^{n}\left(\hat{v}_i^n\right)^{4/3}\left(\hat{v}_{i-1}^n\right)^{4/3}\left(\hat{v}_{i-2}^n\right)^{4/3} \\
&\quad + n^2\sum_{i=4}^{n}\left(\hat{v}_i^n\right)^{2/3}\left(\hat{v}_{i-1}^n\right)^{4/3}\left(\hat{v}_{i-2}^n\right)^{4/3}\left(\hat{v}_{i-3}^n\right)^{2/3} \\
&\quad + n^2\sum_{i=5}^{n}\left(\hat{v}_i^n\right)^{2/3}\left(\hat{v}_{i-1}^n\right)^{2/3}\left(\hat{v}_{i-2}^n\right)^{4/3}\left(\hat{v}_{i-3}^n\right)^{2/3}\left(\hat{v}_{i-4}^n\right)^{2/3}\Bigg),
\end{aligned}
$$

for some constant C that does not depend on n. The desired result follows from Condition A(i).

Proof of Theorem 3.2. Strong asymptotic size control: Since $T_n \xrightarrow{st} N(0,1)$, in restriction to Ω_0, for all measurable subsets S of Ω_0, we have $P\left(T_n \leq x | S\right) \to \Phi(x)$, as $n \to \infty$, where $\Phi(x)$ is the cumulative distribution function of the standard normal random variable. Also, since the bootstrap is valid on Ω_0, in restriction to this set, we have $P^*(T_n^* \leq x) \xrightarrow{P} \Phi(x)$. Thus, by continuity of $\Phi(\cdot)$, $\sup_{x\in\mathbb{R}}\left|P^*(T_n^* \leq x) - P(T_n \leq x|S)\right| \xrightarrow{P} 0$. As a result, letting $q_{n,1-\alpha}^*$ denote the bootstrap $(1-\alpha)$-quantile, we have $P(T_n > q_{n,1-\alpha}^*|S) \xrightarrow{P} \alpha$. This establishes that the bootstrap test controls the strong asymptotic size.

Alternative-consistency: Since in restriction to Ω_1 we still have under Condition A that $T_n^* \xrightarrow{d^*} N(0,1)$, in prob-$P$, we have $T_n^* = O_{P^*}(1)$, in prob-P. As a result, we can claim that $q_{n,1-\alpha}^* = O_P(1)$. Since $T_n \xrightarrow{P} +\infty$ on Ω_1, it is clear that $P(\left\{T_n \leq q_{n,1-\alpha}^*\right\} \cap \Omega_1) \to 0 \quad \text{as} \quad n \to \infty$. This establishes the alternative-consistency of the bootstrap test.

To prove Lemma 3.2, we rely on the following auxiliary result, the proof of which is omitted since it follows from simple algebra.

Lemma S1.2 *Let $\{a_i : i = 1, \ldots, n\}$ be any sequence such that for $i = 1, \ldots, n/M$, $a_{j+(i-1)M} = \bar{a}_i$, $j = 1, \ldots, M$. Then, for any $(s_1, \ldots, s_K) \in \mathbb{R}^K$, letting $s = \sum_{k=1}^{K} s_k$ and $\bar{s}_k = \sum_{l=1}^{k} s_l$, we have that for $M \geq K-1$,*

$$\sum_{i=1}^{n}\prod_{k=1}^{K} a_{i-k+1}^{s_k} = (M-K+1)\sum_{j=1}^{n/M}\left(\bar{a}_j\right)^{s} + \sum_{k=1}^{K-1}\sum_{j=2}^{n/M}\left(\bar{a}_j\right)^{\bar{s}_k}\left(\bar{a}_{j-1}\right)^{s-\bar{s}_k}.$$

Proof of Lemma 3.2. For $i = 1, \ldots, \frac{n}{k_n}$ and $j = 1, \ldots, k_n$, let us denote $\hat{v}_{j+(i-1)k_n}^n$ by $\bar{v}_i$. For k_n large enough, by Lemma S1.2, we have

$$n^{-1+\frac{q}{2}}\sum_{i=K}^{n}\prod_{k=1}^{K}\left(\hat{v}_{i-k+1}^n\right)^{\frac{q_k}{2}} = \frac{k_n}{n}\sum_{i=1}^{n/k_n}(n\bar{v}_i)^{\frac{q}{2}} + (1-K)\frac{1}{n}\sum_{i=1}^{n/k_n}(n\bar{v}_i)^{\frac{q}{2}} + \frac{1}{n}\sum_{k=1}^{K-1}\sum_{i=2}^{n/k_n}(n\bar{v}_i)^{\frac{\bar{q}_k}{2}}(n\bar{v}_{i-1})^{\frac{q-\bar{q}_k}{2}}.$$

Using the notations of Theorem A.1, note that $n\bar{v}_i = \hat{c}_{i,n}$. Hence, by this theorem,

$$\frac{k_n}{n}\sum_{i=1}^{n/k_n}(n\bar{v}_i)^{\frac{q}{2}} = \frac{k_n}{n}\sum_{i=1}^{n/k_n}(\hat{c}_{i,n})^{\frac{q}{2}} \xrightarrow{P} \int_0^1 \sigma_u^q du.$$

This also shows that

$$\frac{1}{n}\sum_{i=1}^{n/k_n}(n\bar{v}_i)^{\frac{q}{2}} = O_P(k_n^{-1}) = o_P(1).$$

Thus, to conclude, it remains to show that, for any $k = 1, \dots, K-1$,

$$\frac{1}{n}\sum_{i=2}^{n/k_n}(n\bar{v}_i)^{\frac{\bar{q}_k}{2}}(n\bar{v}_{i-1})^{\frac{q-\bar{q}_k}{2}} \equiv \frac{1}{n}\sum_{i=2}^{n/k_n}(\hat{c}_{i,n})^{\frac{\bar{q}_k}{2}}(\hat{c}_{i-1,n})^{\frac{q-\bar{q}_k}{2}} = o_P(1).$$

For $x, y \in \mathbb{R}$, let $g(x,y) = |x|^{\frac{\bar{q}_k}{2}}|y|^{\frac{q-\bar{q}_k}{2}}$. We have that

$$|g(x,y)| \leq \max\left(1, (|x|+|y|)^{\frac{q}{2}}\right) \leq 1 + (|x|+|y|)^{\frac{q}{2}} \leq 1 + C_q\left(|x|^{\frac{q}{2}} + |y|^{\frac{q}{2}}\right) \leq C_q\left(1 + |x|^{\frac{q}{2}} + |y|^{\frac{q}{2}}\right)$$

for some $C_q \geq 1$ where the third inequality follows from the C_r-inequality. Given Theorem A.1,

$$\frac{k_n}{n}\sum_{i=2}^{n/k_n} g(\hat{c}_{i,n}, \hat{c}_{i-1,n}) \xrightarrow{P} \int_0^1 \sigma_u^q du,$$

hence

$$\frac{1}{n}\sum_{i=2}^{n/k_n} g(\hat{c}_{i,n}, \hat{c}_{i-1,n}) = O_P(k_n^{-1}) = o_P(1).$$

Proof of Theorem 3.3. It suffices to verify Condition A(i) and A(ii). Take Condition A(i). If X is continuous, by Lemma 3.2, A(i) holds for all $q \in \mathbb{R}_+$ and in particular for $q \in [0,8]$. If X is not continuous, let $\bar{q} = 8$ and $0 \leq q \leq \bar{q}$. If $q < 2$, the convergence statement in A(i) holds, given Lemma 3.2. If $2 \leq q \leq \bar{q}$, since $q \mapsto (q-1)/(2q-r)$ is an increasing function on $[2, \bar{q}]$, $\varpi \geq \frac{7}{16-r} = \frac{\bar{q}-1}{2\bar{q}-r} \geq \frac{q-1}{2q-r}$, and Lemma 3.2 implies the convergence statement in A(i). Next, consider Condition A(ii). If X is continuous, given Lemma 1 of Barndorff-Nielsen, Shephard and Winkel (2006), $|r_i| = O_P(\sqrt{(\log(n))/n})$, uniformly over $i = 1, \dots, n$. Thus,

$$n\sum_{j=1}^{[n/(L_n+1)]}\left(\hat{v}^n_{j(L_n+1)}\right)^2 = O_P(n^{-\alpha}(\log(n))^2) = o_P(1),$$

for all $\alpha \in (0, \frac{3}{7})$. Hence, A(ii) is fulfilled. If X is not continuous, thanks to the truncation, we have that

$$n\sum_{j=1}^{[n/(L_n+1)]}\left(\hat{v}^n_{j(L_n+1)}\right)^2 = O_P\left(n^{2-\alpha}u_n^4\right) = O_P\left(n^{2-\alpha-4\varpi}\right).$$

Note that

$$2 - 4\varpi \leq \frac{4-2r}{16-r} \leq \frac{2}{7}.$$

Hence, A(ii) is fulfilled as we can choose $\alpha \in (\frac{2}{7}, \frac{3}{7})$.

Appendix S2: Asymptotic expansions of the cumulants of T_n, T_n^* and $\bar{T}_n^*$

In this section, we provide proofs for the results in Section 4. We start by introducing some notations and by presenting alternative expressions of T_n, T_n^* and $\bar{T}_n^*$ that are suitable for higher order expansions. Then, we provide proofs of the main theorems, followed by useful auxiliary lemmas along with their proofs.

We let $v_i^n = \int_{(i-1)/n}^{i/n} \sigma_u^2 du$, $\overline{\sigma^q} \equiv \int_0^1 \sigma_u^q du$ and $\sigma_{q,p} \equiv \frac{\overline{\sigma^q}}{\left(\overline{\sigma^p}\right)^{q/p}}$, for any $q, p > 0$. Throughout this section, $E(\cdot)$ and $Var(\cdot)$ denote expectation and variance of the relevant quantities conditionally on the volatility process σ.

We rely on the following expression of the test statistic T_n:

$$T_n = (S_n + A_n) \left(\frac{\hat{V}_n}{V_n} \right)^{-1/2} = (S_n + A_n) \left(1 + \frac{1}{\sqrt{n}} (U_n + B_n) \right)^{-1/2}, \tag{S2.1}$$

where

$$\begin{aligned}
S_n &= S_{n,1} - S_{n,2} \equiv \tfrac{\sqrt{n}(RV_n - E(RV_n))}{\sqrt{V_n}} - \tfrac{\sqrt{n}(BV_n - E(BV_n))}{\sqrt{V_n}} \\
A_n &= \tfrac{\sqrt{n}(E(RV_n) - E(BV_n))}{\sqrt{V_n}} = \tfrac{\sqrt{n}}{\sqrt{V_n}} \left(\sum_{i=1}^n v_i^n - \sum_{i=2}^n \left| v_{i-1}^n \right|^{1/2} \left| v_i^n \right|^{1/2} \right) \\
U_n &= \tfrac{\sqrt{n}\left(\hat{V}_n - E\left(\hat{V}_n\right)\right)}{V_n} \\
\hat{V}_n &= \tau \tfrac{n}{k_{\frac{4}{3}}^3} \sum_{i=3}^n |r_i|^{4/3} \left| r_{i-1} \right|^{4/3} \left| r_{i-2} \right|^{4/3} \\
E\left(\hat{V}_n\right) &= \tau n \sum_{i=3}^n \left| v_{i-2}^n \right|^{2/3} \left| v_{i-1}^n \right|^{2/3} \left| v_i^n \right|^{2/3}; \quad \tau = \theta - 2 = \left(k_1^{-4} - 1 \right) + 2 \left(k_1^{-2} - 1 \right) - 2 \\
V_n &= Var\left(\sqrt{n} \left(RV_n - BV_n \right) \right) \\
&= 2n \sum_{i=1}^n \left(v_i^n \right)^2 - 2 \left[n \sum_{i=2}^n \left(v_{i-1}^n \right)^{1/2} \left(v_i^n \right)^{3/2} + n \sum_{i=2}^n \left(v_{i-1}^n \right)^{3/2} \left(v_i^n \right)^{1/2} \right] \\
&\quad + \left(k_1^{-4} - 1 \right) n \sum_{i=2}^n \left(v_{i-1}^n \right) \left(v_i^n \right) + 2 \left(k_1^{-2} - 1 \right) n \sum_{i=3}^n \left(v_{i-2}^n \right)^{1/2} \left(v_{i-1}^n \right) \left(v_i^n \right)^{1/2} \\
B_n &= \tfrac{\sqrt{n}\left(E\left(\hat{V}_n\right) - V_n\right)}{V_n} = \tfrac{n^{3/2}}{V_n} \tau \sum_{i=3}^n \left| v_{i-2}^n \right|^{2/3} \left| v_{i-1}^n \right|^{2/3} \left| v_i^n \right|^{2/3} \\
&\quad - 2 \tfrac{n^{3/2}}{V_n} \left[\sum_{i=1}^n \left(v_i^n \right)^2 - \sum_{i=2}^n \left(v_{i-1}^n \right)^{1/2} \left(v_i^n \right)^{3/2} - \sum_{i=2}^n \left(v_{i-1}^n \right)^{3/2} \left(v_i^n \right)^{1/2} \right] \\
&\quad - \tfrac{n^{3/2}}{V_n} \left[\left(k_1^{-4} - 1 \right) \sum_{i=2}^n \left(v_{i-1}^n \right) \left(v_i^n \right) + 2 \left(k_1^{-2} - 1 \right) \sum_{i=3}^n \left(v_{i-2}^n \right)^{1/2} \left(v_{i-1}^n \right) \left(v_i^n \right)^{1/2} \right].
\end{aligned}$$

Similarly, for the bootstrap statistics, we have:

$$T_n^* = \frac{\sqrt{n} \left(RV_n^* - BV_n^* - E^* \left(RV_n^* - BV_n^* \right) \right)}{\sqrt{\hat{V}_n^*}} = \left(S_n^* + A_n^* \right) \left(1 + \frac{1}{\sqrt{n}} \left(U_n^* + B_n^* \right) \right)^{-1/2} \tag{S2.2}$$

and

$$\bar{T}_n^* = \frac{\sqrt{n}\left(RV_n^* - BV_n^* - E^*\left(RV_n^* - BV_n^*\right)\right)}{\sqrt{\hat{V}_n^*}} + \frac{1}{2}\frac{\sqrt{n}\left(\hat{v}_1^n + \hat{v}_n^n\right)}{\sqrt{\hat{V}_n^*}} = \left(S_n^* + \bar{A}_n^*\right)\left(1 + \frac{1}{\sqrt{n}}\left(U_n^* + B_n^*\right)\right)^{-1/2}, \tag{S2.3}$$

where:

$$\begin{aligned}
S_n^* &= S_{n,1}^* - S_{n,2}^* \equiv \tfrac{\sqrt{n}(RV_n^* - E^*(RV_n^*))}{\sqrt{V_n^*}} - \tfrac{\sqrt{n}(BV_n^* - E^*(BV_n^*))}{\sqrt{V_n^*}} \\
A_n^* &= 0 \\
\bar{A}_n^* &= \tfrac{1}{2}\tfrac{\sqrt{n}\left(\hat{v}_1^n + \hat{v}_n^n\right)}{\sqrt{V_n^*}} \\
U_n^* &= \tfrac{\sqrt{n}\left(\hat{V}_n^* - E^*\left(\hat{V}_n^*\right)\right)}{V_n^*} \\
\hat{V}_n^* &= \tau \tfrac{n}{k_{\frac{4}{3}}^3} \sum_{i=3}^{n} \left|r_{i-2}^*\right|^{4/3} \left|r_{i-1}^*\right|^{4/3} \left|r_i^*\right|^{4/3} \\
E^*\left(\hat{V}_n^*\right) &= \tau n \sum_{i=3}^{n} \left(\hat{v}_{i-2}^n\right)^{2/3} \left(\hat{v}_{i-1}^n\right)^{2/3} \left(\hat{v}_i^n\right)^{2/3} \\
V_n^* &= Var^*\left(\sqrt{n}\left(RV_n^* - BV_n^*\right)\right) \\
&= \sum_{i=1}^{n} \left(\hat{v}_i^n\right)^2 + (k_1^{-4} - 1) n \sum_{i=2}^{n} \left(\hat{v}_i^n\right)\left(\hat{v}_{i-1}^n\right) + 2(k_1^{-2} - 1) n \sum_{i=3}^{n} \left(\hat{v}_i^n\right)^{1/2} \left(\hat{v}_{i-1}^n\right) \left(\hat{v}_{i-2}^n\right)^{1/2} \\
&\quad -2n \sum_{i=2}^{n} \left(\hat{v}_i^n\right)^{3/2} \left(\hat{v}_{i-1}^n\right)^{1/2} - 2n \sum_{i=2}^{n} \left(\hat{v}_i^n\right)^{1/2} \left(\hat{v}_{i-1}^n\right)^{3/2} \\
B_n^* &= \tfrac{\sqrt{n}\left(E^*\left(\hat{V}_n^*\right) - V_n^*\right)}{V_n^*} = \tfrac{n^{3/2}}{V_n^*} \tau \sum_{i=3}^{n} \left|\hat{v}_{i-2}^n\right|^{2/3} \left|\hat{v}_{i-1}^n\right|^{2/3} \left|\hat{v}_i^n\right|^{2/3} \\
&\quad -2\tfrac{n^{3/2}}{V_n^*}\left[\sum_{i=1}^{n} \left(\hat{v}_i^n\right)^2 - \sum_{i=2}^{n} \left(\hat{v}_{i-1}^n\right)^{1/2} \left(\hat{v}_i^n\right)^{3/2} - \sum_{i=2}^{n} \left(\hat{v}_{i-1}^n\right)^{3/2} \left(\hat{v}_i^n\right)^{1/2}\right] \\
&\quad -\tfrac{n^{3/2}}{V_n^*}\left[\left(k_{1,1}^{-4} - 1\right) \sum_{i=2}^{n} \left(\hat{v}_{i-1}^n\right)\left(\hat{v}_i^n\right) + 2\left(k_1^{-2} - 1\right) \sum_{i=3}^{n} \left(\hat{v}_{i-2}^n\right)^{1/2} \left(\hat{v}_{i-1}^n\right) \left(\hat{v}_i^n\right)^{1/2}\right].
\end{aligned}$$

S2.1 Proofs of the main results

Proof of Theorem 4.1. The first and third cumulants of T_n are given by

$$\kappa_1\left(T_n\right) = E\left(T_n\right) \quad \text{and} \quad \kappa_3\left(T_n\right) = E\left(T_n^3\right) - 3E\left(T_n^2\right)E\left(T_n\right) + 2\left[E\left(T_n\right)\right]^3.$$

Following Gonçalves and Meddahi (2009), provided that these two cumulants exist, we identify the terms of order up to $O\left(n^{-1/2}\right)$ in their asymptotic expansions. We first derive the first three moments of T_n up to $O\left(n^{-1/2}\right)$. For a given value k, a first-order Taylor expansion of $f(x) = (1+x)^{-k/2}$ around 0 yields $f(x) = 1 - \frac{k}{2}x + O\left(x^2\right)$. We first derive the moments of T_n up to $O\left(n^{-1/2}\right)$. Using Lemmas S2.1 and S2.3, we have $A_n = O\left(n^{-1/2}\right)$ and $B_n = O(1)$. Thus, using (S2.1), we have:

$$\begin{aligned}
T_n^k &= \left(S_n + A_n\right)^k - \frac{k}{2\sqrt{n}}\left(S_n + A_n\right)^k\left(U_n + B_n\right) + O_P\left(n^{-1}\right) \\
&\equiv \widetilde{T}_n^k + O_P\left(n^{-1}\right).
\end{aligned}$$

Hence, for $k = 1, 2, 3$, the moments of $\widetilde{T}_n^k$ are given by

$$
\begin{aligned}
E\left(\widetilde{T}_n\right) &= E\left(S_n + A_n\right) - \frac{1}{2\sqrt{n}} E[\left(S_n + A_n\right)\left(U_n + B_n\right)] \\
&= E\left(S_n\right) + A_n - \frac{1}{2\sqrt{n}}\left[E\left(S_n U_n\right) + B_n E\left(S_n\right) + A_n E\left(U_n\right) + A_n B_n\right] \\
&= -\frac{E\left(S_n U_n\right)}{2\sqrt{n}} + \underbrace{A_n}_{\equiv b_{1,n}} + O\left(n^{-1}\right),
\end{aligned}
$$

$$
\begin{aligned}
E\left(\widetilde{T}_n^2\right) &= \left(S_n + A_n\right)^2 - \frac{1}{\sqrt{n}} E[\left(S_n + A_n\right)^2\left(U_n + B_n\right)] \\
&= E\left(S_n^2\right) + 2A_n E\left(S_n\right) - \frac{1}{\sqrt{n}} E\left(S_n^2 U_n\right) - E\left(S_n^2\right)\frac{B_n}{\sqrt{n}} + O\left(n^{-1}\right) \\
&= 1 - \frac{1}{\sqrt{n}} E\left(S_n^2 U_n\right) \underbrace{-\frac{B_n}{\sqrt{n}}}_{\equiv b_{2,n}} + O\left(n^{-1}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
E\left(\widetilde{T}_n^3\right) &= E\left(S_n + A_n\right)^3 - \frac{3}{2\sqrt{n}} E[\left(S_n + A_n\right)^3\left(U_n + B_n\right)] \\
&= E\left(S_n^3 + 3A_n S_n^2\right) - \frac{3}{2\sqrt{n}} E[S_n^3\left(U_n + B_n\right)] + O(n^{-1}) \\
&= E\left(S_n^3\right) - \frac{3}{2\sqrt{n}} E\left(S_n^3 U_n\right) + 3A_n E\left(S_n^2\right) - \frac{3}{2\sqrt{n}} E\left(B_n S_n^3\right) + O\left(n^{-1}\right) \\
&= E\left(S_n^3\right) - \frac{3}{2\sqrt{n}} E\left(S_n^3 U_n\right) + \underbrace{3A_n - \frac{3}{2}\frac{B_n}{\sqrt{n}} E\left(S_n^3\right)}_{\equiv b_{3,n}} + O\left(n^{-1}\right)
\end{aligned}
$$

where we used $E\left(S_n\right) = 0$ and $E\left(S_n^2\right) = 1$ (see Lemma S2.5 in the next subsection). Below, we let

$$
b_{1,n} = A_n,\ b_{2,n} = -\frac{B_n}{\sqrt{n}},\ \text{and } b_{3,n} = 3A_n - \frac{3}{2}\frac{B_n}{\sqrt{n}} E\left(S_n^3\right).
$$

It follows that

$$
\kappa_1\left(T_n\right) = -\frac{E\left(S_n U_n\right)}{2\sqrt{n}} + b_{1,n}, \tag{S2.4}
$$

$$
\begin{aligned}
\kappa_3\left(T_n\right) &= E\left(S_n^3\right) - \frac{3}{2\sqrt{n}} E\left(S_n^3 U_n\right) + b_{3,n} + 2\left[b_{1,n}^3 - 3b_{1,n}^2 \frac{E\left(S_n U_n\right)}{2\sqrt{n}} + 3b_{1,n}\frac{\left(E\left(S_n U_n\right)\right)^2}{4n} - \frac{\left(E\left(S_n U_n\right)\right)^3}{8n^{3/2}}\right] \\
&\quad -3\left[b_{1,n} b_{2,n} - b_{2,n}\frac{E\left(S_n U_n\right)}{2\sqrt{n}} - \frac{b_{1,n}}{\sqrt{n}} E\left(S_n^2 U_n\right) + \frac{E\left(S_n U_n\right) E\left(S_n^2 U_n\right)}{2n} - \frac{E\left(S_n U_n\right)}{2\sqrt{n}} + b_{1,n}\right] \\
&= \kappa_{3,1}\left(T_n\right) + \kappa_{3,2}\left(T_n\right),
\end{aligned} \tag{S2.5}
$$

where

$$\begin{aligned}\kappa_{3,1}(T_n) &= E\left(S_n^3\right) + \frac{3}{2}\frac{E(S_nU_n)}{\sqrt{n}} - \frac{3}{2\sqrt{n}}E\left(S_n^3U_n\right) - 3\left[\frac{E(S_nU_n)E\left(S_n^2U_n\right)}{2n}\right] - \frac{(E(S_nU_n))^3}{4n^{3/2}}, \text{ and}\\ \kappa_{3,2}(T_n) &= b_{3,n} - 3b_{1,n} - 3\left[b_{1,n}b_{2,n} - b_{2,n}\frac{E(S_nU_n)}{2\sqrt{n}} - \frac{b_{1,n}}{\sqrt{n}}E\left(S_n^2U_n\right)\right]\\ &\quad +2\left[b_{1,n}^3 - 3b_{1,n}^2\frac{E(S_nU_n)}{2\sqrt{n}} + 3b_{1,n}\frac{(E(S_nU_n))^2}{4n}\right]\end{aligned}$$

Therefore, from Lemmas S2.5(a2) and S2.3, we can write

$$\kappa_1(T_n) = \frac{1}{\sqrt{n}}\kappa_1 + o\left(\frac{1}{\sqrt{n}}\right).$$

with

$$\kappa_1 = \kappa_{1,1}+\kappa_{1,2}, \quad \kappa_{1,1} = \lim_{n\to\infty}\sqrt{n}b_{1,n} = \frac{\sigma_0^2+\sigma_1^2}{2\sqrt{\tau\int_0^1\sigma_u^4du}} = \frac{\sigma_0^2+\sigma_1^2}{2\sqrt{\tau\overline{\sigma^4}}}, \quad \kappa_{1,2} = \lim_{n\to\infty}\left[-\frac{E(S_nU_n)}{2}\right] = -\frac{a_1}{2}\sigma_{6,4},$$

where a_1 is defined as in Lemma S2.5(a2). Similarly, for the third cumulant, we have

$$\kappa_3(T_n) = \frac{1}{\sqrt{n}}\kappa_3 + o\left(\frac{1}{\sqrt{n}}\right),$$

where

$$\kappa_3 = \kappa_{3,1} + \kappa_{3,2},$$

such that

$$\begin{aligned}\kappa_{3,1} &= \lim_{n\to\infty}\sqrt{n}\kappa_{3,1}(T_n)\\ &= \lim_{n\to\infty}\sqrt{n}E\left(S_n^3\right) + \frac{3}{2}\lim_{n\to\infty}E(S_nU_n) - \frac{3}{2}\lim_{n\to\infty}E\left(S_n^3U_n\right)\\ &= \left[a_2 + \frac{3}{2}(a_1 - a_3)\right]\sigma_{6,4},\end{aligned}$$

with a_2, a_1 and a_3 given in Lemma S2.5. The other terms in $\sqrt{n}\kappa_{3,1}(T_n)$ have zero limit:

$$\kappa_{3,2} = p\lim_{n\to\infty}\sqrt{n}\kappa_{3,2}(T_n) = 3\kappa_{1,2} - 3\kappa_{1,2} = 0,$$

where we use in this derivation Lemma S2.5 and the fact that $A_n = O\left(n^{-1/2}\right)$ and $B_n = O(1)$.

Proof of Theorem 4.2. So long as $A_n^* = O_P(n^{-1/2})$ and $B_n^* = O_P(1)$, we can use the same arguments as in the proof of Theorem 4.1 and claim that

$$\kappa_1^*(T_n^*) = -\frac{E^*(S_n^*U_n^*)}{2\sqrt{n}} + b_{1,n}^*, \tag{S2.6}$$

$$\kappa_3^*(T_n^*) = \kappa_{3,1}^*(T_n^*) + \kappa_{3,2}^*(T_n^*), \text{ where} \tag{S2.7}$$

$$\begin{aligned}\kappa_{3,1}^*(T_n^*) &= E^*\left(S_n^{*3}\right) + \frac{3}{2}\frac{E^*(S_n^*U_n^*)}{\sqrt{n}} - \frac{3}{2\sqrt{n}}E^*\left(S_n^{*3}U_n^*\right) - 3\left[\frac{E^*(S_n^*U_n^*)E^*\left(S_n^{*2}U_n^*\right)}{2n}\right] - \frac{(E^*(S_n^*U_n^*))^3}{4n^{3/2}}, \text{ and}\\ \kappa_{3,2}^*(T_n^*) &= b_{3,n}^* - 3b_{1,n}^* - 3\left[b_{1,n}^*b_{2,n}^* - b_{2,n}^*\frac{E^*(S_n^*U_n^*)}{2\sqrt{n}} - \frac{b_{1,n}^*}{\sqrt{n}}E^*\left(S_n^{*2}U_n^*\right)\right]\\ &\quad +2\left[b_{1,n}^{*3} - 3b_{1,n}^{*2}\frac{E^*(S_n^*U_n^*)}{2\sqrt{n}} + 3b_{1,n}^*\frac{(E^*(S_n^*U_n^*))^2}{4n}\right],\end{aligned}$$

with

$$b_{1,n}^* = A_n^* = 0,\ \ b_{2,n}^* = -\frac{B_n^*}{\sqrt{n}} \text{ and } b_{3,n}^* = 3A_n^* - \frac{3}{2}\frac{B_n^*}{\sqrt{n}}E^*\left(S_n^{*3}\right).$$

We can write:

$$\kappa_1^*\left(T_n^*\right) = \frac{1}{\sqrt{n}}\kappa_1^* + o_P\left(\frac{1}{\sqrt{n}}\right) \text{ and } \kappa_3^*\left(T_n^*\right) = \frac{1}{\sqrt{n}}\kappa_3^* + o_P\left(\frac{1}{\sqrt{n}}\right).$$

By Lemma S2.6, we have

$$\kappa_1^* = p\lim_{n\to\infty}\sqrt{n}\kappa_1^*\left(T_n^*\right) = \kappa_{1,2} \neq \kappa_1$$

and

$$\kappa_3^* = p\lim_{n\to\infty}\sqrt{n}\kappa_{3,1}^*\left(T_n^*\right) + p\lim_{n\to\infty}\sqrt{n}\kappa_{3,2}^*\left(T_n^*\right) = \kappa_{3,1} + \kappa_{3,2} = \kappa_3.$$

We recall that $A_n^* = 0 = O_P(n^{-1/2})$ and Lemma S2.6(a6) ensures that $B_n^* = O_P(1)$, which concludes the proof.

Proof of Theorem 4.3. From Theorem 9.3.2 of Jacod and Protter (2012), we have that

$$p\lim_{n\to\infty} n\hat{v}_1^n = \sigma_0^2 \quad \textit{and} \quad p\lim_{n\to\infty} n\hat{v}_n^n = \sigma_1^2$$

showing that $\bar{A}_n^* = O_P(n^{-1/2})$. Using the same arguments as in the proof of Theorem 4.2, it follows that $\kappa_1^*\left(\bar{T}_n^*\right)$ and $\kappa_3^*\left(\bar{T}_n^*\right)$ are given as in (S2.6) and (S2.7), respectively, where we now set

$$b_{1,n}^* = \bar{A}_n^* = \frac{1}{2}\frac{\sqrt{n}\left(\hat{v}_1^n + \hat{v}_n^n\right)}{\sqrt{V_n^*}},\ \ b_{2,n}^* = -\frac{B_n^*}{\sqrt{n}} \text{ and } b_{3,n}^* = 3\bar{A}_n^* - \frac{3}{2}\frac{B_n^*}{\sqrt{n}}E^*\left(S_n^{*3}\right).$$

Letting $\overline{\kappa}_1^* = p\lim_{n\to\infty}\sqrt{n}\kappa_1^*\left(\bar{T}_n^*\right)$ and $\overline{\kappa}_3^* = p\lim_{n\to\infty}\sqrt{n}\kappa_3^*\left(\bar{T}_n^*\right)$ $\overline{\kappa}_{3n}^*$, we have:

$$\kappa_1^*\left(\bar{T}_n^*\right) = \frac{1}{\sqrt{n}}\overline{\kappa}_1^* + o_P\left(\frac{1}{\sqrt{n}}\right) \text{ and } \kappa_3^*\left(\bar{T}_n^*\right) = \frac{1}{\sqrt{n}}\overline{\kappa}_3^* + o_P\left(\frac{1}{\sqrt{n}}\right).$$

Using the expansions in (S2.6) and (S2.7), Lemma S2.6 and the fact that $p\lim_{n\to\infty}\sqrt{n}b_{1,n}^* = \frac{\sigma_0^2+\sigma_1^2}{2\sqrt{\overline{\tau\sigma^4}}}$, we can conclude that

$$\overline{\kappa}_1^* = \kappa_{1,1} + \kappa_{1,2} = \kappa_1 \quad \textit{and} \quad \overline{\kappa}_3^* = \kappa_{3,1} + \kappa_{3,2} = \kappa_3.$$

S2.2 Auxiliary lemmas

Lemma S2.1 *If the volatility process σ is càdlàg and locally bounded away from 0 and $\int_0^t \sigma_u^2 du < \infty$ for all $t < \infty$, then, for any $q_1, q_2, q_3 \geq 0$, we have that*

$$n^{-1+q_1+q_2+q_3}\left(\sum_{i=1}^{n-2}(v_i^n)^{q_1}(v_{i+1}^n)^{q_2}(v_{i+2}^n)^{q_3} - \sum_{i=1}^{n}(v_i^n)^{q_1+q_2+q_3}\right) = O_P(n^{-1/2}).$$

Lemma S2.2 *If the volatility process σ is càdlàg bounded away from zero and $\int_0^t \sigma_u^2 du < \infty$ for all $t < \infty$, then for any $q_1, q_2, q_3 \geq 0$, such that $q \equiv q_1 + q_2 + q_3 > 0$, as $n \to \infty$, we have that*

$$n^{-1+q/2}\sum_{i=K}^{n}\prod_{k=1}^{K}\left(v_{i-k+1}^n\right)^{q_k/2} \xrightarrow{p} \overline{\sigma^q} > 0, \tag{S2.8}$$

with $K \in \{1, 2, 3\}$.

Lemma S2.3 *If Assumption V holds, then, as $n \to \infty$,*

$$n\left(\sum_{i=1}^{n} v_i^n - \sum_{i=2}^{n} (v_i^n)^{1/2}(v_{i-1}^n)^{1/2}\right) \xrightarrow{p} \frac{1}{2}(\sigma_0^2 + \sigma_1^2),$$

with $v_i^n = \int_{\frac{i-1}{n}}^{\frac{i}{n}} \sigma_u^2 du$.

Lemma S2.4 *Let X_t be described as in (10). Then, conditionally on the path of volatility, for $i = 1, \ldots, n$, $r_i \sim N\left(0, v_i^n\right)$, where $v_i^n = \int_{(i-1)/n}^{i/n} \sigma_u^2 du$ and the following results hold:*

(a1)

$$E\left(S_{n,1}\right) = 0 \text{ and } E\left(S_{n,2}\right) = 0.$$

(a2)

$$E\left(S_{n,1}U_n\right) = \frac{\tau\left(k_{\frac{4}{3}}^2 k_{\frac{10}{3}} - k_{\frac{4}{3}}^3\right)}{k_{\frac{4}{3}}^3 V_n^{3/2}} n^2 \left[\begin{array}{l} \sum_{i=3}^{n} \left(v_{i-2}^n\right)^{2/3} \left(v_{i-1}^n\right)^{2/3} \left(v_i^n\right)^{5/3} \\ +\sum_{i=3}^{n} \left(v_{i-2}^n\right)^{5/3} \left(v_{i-1}^n\right)^{2/3} \left(v_i^n\right)^{2/3} \\ +\sum_{i=3}^{n} \left(v_{i-2}^n\right)^{2/3} \left(v_{i-1}^n\right)^{5/3} \left(v_i^n\right)^{2/3} \end{array}\right].$$

(a3)

$$\begin{aligned} E\left(S_{n,2}U_n\right) &= \frac{\tau\left(k_{\frac{4}{3}} k_{\frac{7}{3}}^2 - k_1^2 k_{\frac{4}{3}}^3\right)}{k_1^2 k_{\frac{4}{3}}^3 V_n^{3/2}} n^2 \left[\begin{array}{l} \sum_{i=3}^{n} \left(v_{i-2}^n\right)^{2/3} \left(v_{i-1}^n\right)^{7/6} \left(v_i^n\right)^{7/6} \\ +\sum_{i=3}^{n} \left(v_{i-2}^n\right)^{7/6} \left(v_{i-1}^n\right)^{7/6} \left(v_i^n\right)^{2/3} \end{array}\right] \\ &+ \frac{\tau\left(k_1 k_{\frac{4}{3}}^2 k_{\frac{7}{3}} - k_1^2 k_{\frac{4}{3}}^3\right)}{k_1^2 k_{\frac{4}{3}}^3 V_n^{3/2}} n^2 \left[\begin{array}{l} \sum_{i=4}^{n} \left(v_{i-3}^n\right)^{1/2} \left(v_{i-2}^n\right)^{7/6} \left(v_{i-1}^n\right)^{2/3} \left(v_i^n\right)^{2/3} \\ +\sum_{i=4}^{n} \left(v_{i-3}^n\right)^{2/3} \left(v_{i-2}^n\right)^{2/3} \left(v_{i-1}^n\right)^{7/6} \left(v_i^n\right)^{1/2} \end{array}\right]. \end{aligned}$$

(a4)

$$E\left(S_{n,1}^2 U_n\right) = O(n^{-1/2}).$$

(a5)

$$E\left(S_{n,1} S_{n,2} U_n\right) = O(n^{-1/2}).$$

(a6)

$$E\left(S_{n,2}^2 U_n\right) = O(n^{-1/2}).$$

(a7)

$$E\left(S_{n,1}^3\right) = \frac{(k_6 - 3k_4 + 2)}{V_n^{3/2}} n^{3/2} \sum_{i=1}^{n} \left(v_i^n\right)^3.$$

(a8)

$$\begin{aligned} E\left(S_{n,1}^2 S_{n,2}\right) &= \frac{\left(k_1 k_5 - k_1^2 k_4 - 2k_1 k_3 + 2k_1^2\right)}{k_1^2 V_n^{3/2}} n^{3/2} \left[\sum_{i=2}^{n} \left(v_{i-1}^n\right)^{1/2} \left(v_i^n\right)^{5/2} + \sum_{i=2}^{n} \left(v_{i-1}^n\right)^{5/2} \left(v_i^n\right)^{1/2}\right] \\ &+ 2\frac{\left(k_3^2 - 2k_1 k_3 + k_1^2\right)}{k_1^2 V_n^{3/2}} n^{3/2} \sum_{i=2}^{n} \left(v_{i-1}^n\right)^{3/2} \left(v_i^n\right)^{3/2}. \end{aligned}$$

(a9)

$$\begin{aligned}
E\left(S_{n,1}S_{n,2}^2\right) &= \frac{2\left(1-k_1^3k_3+k_1^4\right)}{k_1^4V_n^{3/2}}n^{3/2}\sum_{i=2}^{n}\left[\left(v_{i-1}^n\right)\left(v_i^n\right)^2+\left(v_{i-1}^n\right)^2\left(v_i^n\right)\right]\\
&+\frac{2\left(k_1^2-k_1^3k_3+k_1^4\right)}{k_1^4V_n^{3/2}}n^{3/2}\sum_i\left[\left(v_{i-2}^n\right)^{1/2}\left(v_{i-1}^n\right)^2\left(v_i^n\right)^{1/2}+\left(v_i^n\right)^2\left(v_{i-1}^n\right)^{1/2}\left(v_{i+1}^n\right)^{1/2}\right]\\
&+\frac{\left(k_1k_3-k_1^3k_3-k_1^2+k_1^4\right)}{k_1^4V_n^{3/2}}n^{3/2}\sum_i\left[\left(v_{i-2}^n\right)^{1/2}\left(v_{i-1}^n\right)\left(v_i^n\right)^{3/2}+\left(v_{i-2}^n\right)^{3/2}\left(v_{i-1}^n\right)\left(v_i^n\right)^{1/2}\right]\\
&+\frac{\left(k_1^4-k_1^2-k_1^3k_3+k_1k_3\right)}{k_1^4V_n^{3/2}}n^{3/2}\sum_i\left[\left(v_{i-1}^n\right)^{3/2}\left(v_i^n\right)\left(v_{i+1}^n\right)^{1/2}+\left(v_i^n\right)\left(v_{i-1}^n\right)^{1/2}\left(v_{i+1}^n\right)^{3/2}\right].
\end{aligned}$$

(a10)

$$\begin{aligned}
E\left(S_{n,2}^3\right) &= \frac{\left(k_3^2-3k_1^2+2k_1^6\right)}{k_1^6V_n^{3/2}}n^{3/2}\sum_{i=2}^{n}\left(v_{i-1}^n\right)^{3/2}\left(v_i^n\right)^{3/2}\\
&+\frac{2\left(k_1k_3-k_1^2-2k_1^4+2k_1^6\right)}{k_1^6V_n^{3/2}}n^{3/2}\sum_{i=3}^{n}\left[\left(v_{i-2}^n\right)^{1/2}\left(v_{i-1}^n\right)^{3/2}v_i^n+v_{i-2}^n\left(v_{i-1}^n\right)^{3/2}\left(v_i^n\right)^{1/2}\right]\\
&+\frac{\left(2k_1^6-2k_1^4-k_1^2+k_1k_3\right)}{k_1^6V_n^{3/2}}n^{3/2}\sum_i\left[\left(v_{i-2}^n\right)\left(v_{i-1}^n\right)^{3/2}\left(v_i^n\right)^{1/2}+\left(v_{i-1}^n\right)^{1/2}\left(v_i^n\right)^{3/2}\left(v_{i+1}^n\right)\right]\\
&+\frac{6\left(k_1^6-2k_1^4+k_1^2\right)}{k_1^6V_n^{3/2}}n^{3/2}\sum_i\left(v_{i-2}^n\right)^{1/2}\left(v_{i-1}^n\right)\left(v_i^n\right)\left(v_{i+1}^n\right)^{1/2}.
\end{aligned}$$

(a11)

$$E\left(S_{n,1}^3U_n\right)=\frac{\tau n^3}{k_{\frac{4}{3}}^3V_n^{5/2}}\left[\begin{array}{c}3\left(\left(k_4-k_2\right)\left(\sum_{i=1}^n(v_i^n)^2\right)\right)\left(k_{\frac{4}{3}}^2k_{\frac{10}{3}}-k_{\frac{4}{3}}^3\right)\\ \times\left[\begin{array}{c}\sum_{i=3}^n\left(v_{i-2}^n\right)^{2/3}\left(v_{i-1}^n\right)^{2/3}\left(v_i^n\right)^{5/3}\\ +\sum_{i=3}^n\left(v_{i-2}^n\right)^{5/3}\left(v_{i-1}^n\right)^{2/3}\left(v_i^n\right)^{2/3}\\ +\sum_{i=3}^n\left(v_{i-2}^n\right)^{2/3}\left(v_{i-1}^n\right)^{5/3}\left(v_i^n\right)^{2/3}\end{array}\right]\end{array}\right]+O(n^{-1}).$$

(a12)

$$E(S_{n,1}^2S_{n,2}U_n)=\frac{\tau}{k_1^2k_{\frac{4}{3}}^3}\frac{n^3}{V_n^{\frac{5}{2}}}[(\mathbf{1})+(\mathbf{2})],$$

where

$$\begin{aligned}
&(\mathbf{1})=(k_4-k_2)\left(\sum_i(v_i^n)^2\right)\\
\times\quad&\left\{k_{\frac{4}{3}}\left(k_{\frac{7}{3}}^2-k_1^2k_{\frac{4}{3}}^2\right)\left(\sum_i(v_i^n)^{7/6}(v_{i-1}^n)^{7/6}(v_{i-2}^n)^{2/3}+\sum_i(v_i^n)^{2/3}(v_{i-1}^n)^{7/6}(v_{i-2}^n)^{7/6}\right)+\right.\\
&\left.k_1k_{\frac{4}{3}}^2\left(k_{\frac{7}{3}}-k_1k_{\frac{4}{3}}\right)\left(\begin{array}{c}\sum_i(v_i^n)^{2/3}(v_{i-1}^n)^{2/3}(v_{i-2}^n)^{7/6}(v_{i-3}^n)^{1/2}\\ +\sum_i(v_{i+1}^n)^{1/2}(v_i^n)^{7/6}(v_{i-1}^n)^{2/3}(v_{i-2}^n)^{2/3}\end{array}\right)\right\}\\
&+O(n^{-4}),
\end{aligned}$$

and

$$(\mathbf{2}) = 2\times\left\{k_1(k_3-k_1k_2)\left(\sum_i (v_i^n)^{3/2}(v_{i-1}^n)^{1/2}+\sum_i (v_i^n)^{1/2}(v_{i-1}^n)^{3/2}\right)\right\}$$

$$\times\quad\Big\{k_{\frac{4}{3}}^2(k_{\frac{10}{3}}-k_2k_{\frac{4}{3}})$$

$$\times\left(\sum_i (v_i^n)^{5/3}(v_{i-1}^n)^{2/3}(v_{i-2}^n)^{2/3}+\sum_i (v_i^n)^{2/3}(v_{i-1}^n)^{5/3}(v_{i-2}^n)^{2/3}+\sum_i (v_i^n)^{2/3}(v_{i-1}^n)^{2/3}(v_{i-2}^n)^{5/3}\right)\Big\}$$

$$+O(n^{-4}).$$

(a13)

$$E(S_{n,1}S_{n,2}^2U_n)=\frac{\tau}{k_1^4k_{\frac{4}{3}}^3}\frac{n^3}{V_n^{\frac{5}{2}}}[(\mathbf{3})+(\mathbf{4})],$$

where

$$\begin{aligned}(\mathbf{3}) \quad = \quad & \left(k_2^2-k_1^4\right)k_{\frac{4}{3}}^2\left(k_{\frac{10}{3}}-k_2k_{\frac{4}{3}}\right)\sum_i v_i^nv_{i-1}^n \\ & \times\left(\sum_i (v_i^n)^{5/3}(v_{i-1}^n)^{2/3}(v_{i-2}^n)^{2/3}+\sum_i (v_i^n)^{2/3}(v_{i-1}^n)^{5/3}(v_{i-2}^n)^{2/3}+\sum_i (v_i^n)^{2/3}(v_{i-1}^n)^{2/3}(v_{i-2}^n)^{5/3}\right) \\ & +O(n^{-4})\end{aligned}$$

and

$$\begin{aligned}(\mathbf{4}) = \quad & 2\times\left(k_1^2(k_2-k_1^2)\sum_i (v_{i+1}^n)^{1/2}v_i^n(v_{i-1}^n)^{1/2}\right)\times\Big\{k_{\frac{4}{3}}^2(k_{\frac{10}{3}}-k_2k_{\frac{4}{3}}) \\ & \times\left(\sum_i (v_i^n)^{5/3}(v_{i-1}^n)^{2/3}(v_{i-2}^n)^{2/3}+\sum_i (v_i^n)^{2/3}(v_{i-1}^n)^{5/3}(v_{i-2}^n)^{2/3}+\sum_i (v_i^n)^{2/3}(v_{i-1}^n)^{2/3}(v_{i-2}^n)^{5/3}\right)\Big\} \\ & +2\times\left\{k_1(k_3-k_1k_2)\left(\sum_i (v_i^n)^{3/2}(v_{i-1}^n)^{1/2}+\sum_i (v_i^n)^{1/2}(v_{i-1}^n)^{3/2}\right)\right\} \\ & \times\Big\{k_{\frac{4}{3}}(k_{\frac{7}{3}}^2-k_1^2k_{\frac{4}{3}}^2)\left(\sum_i (v_i^n)^{7/6}(v_{i-1}^n)^{7/6}(v_{i-2}^n)^{2/3}+\sum_i (v_{i+1}^n)^{2/3}(v_i^n)^{7/6}(v_{i-1}^n)^{7/6}\right) \\ & +k_1k_{\frac{4}{3}}^2(k_{\frac{7}{3}}-k_1k_{\frac{4}{3}})\left(\begin{array}{c}\sum_i (v_i^n)^{1/2}(v_{i-1}^n)^{7/6}(v_{i-2}^n)^{2/3}(v_{i-3}^n)^{2/3} \\ +\sum_i (v_i^n)^{2/3}(v_{i+1}^n)^{2/3}(v_i^n)^{7/6}(v_{i-1}^n)^{1/2}\end{array}\right)\Big\}.\end{aligned}$$

(a14)

$$E(S_{n,2}^3U_n)=\frac{\tau}{k_1^6k_{\frac{4}{3}}^3}\frac{n^3}{V_n^{\frac{5}{2}}}[(\mathbf{6})+(\mathbf{7})],$$

where

$$(\mathbf{6})+(\mathbf{7}) = 3\times\left\{(k_2^2-k_1^4)\sum_i v_i^n v_{i-1}^n\right\}\times$$

$$\times\quad\left\{k_{\frac{4}{3}}\left(k_{\frac{7}{3}}^2-k_1^2k_{\frac{4}{3}}^2\right)\left(\sum_i (v_i^n)^{7/6}(v_{i-1}^n)^{7/6}(v_{i-2}^n)^{2/3}+\sum_i (v_{i+1}^n)^{2/3}(v_i^n)^{7/6}(v_{i-1}^n)^{7/6}\right)\right.$$

$$\left. +k_1k_{\frac{4}{3}}^2\left(k_{\frac{7}{3}}-k_1k_{\frac{4}{3}}\right)\left(\begin{array}{c}\sum_i (v_i^n)^{1/2}(v_{i-1}^n)^{7/6}(v_{i-2}^n)^{2/3}(v_{i-3}^n)^{2/3} \\ +\sum_i (v_{i+2}^n)^{2/3}(v_{i+1}^n)^{2/3}(v_i^n)^{7/6}(v_{i-1}^n)^{1/2}\end{array}\right)\right\}$$

$$+O(n^{-4}).$$

Lemma S2.5 *Let X_t be described as in (10). Then, conditionally on the path of volatility, for $i = 1,\ldots,n$, $r_i \sim N\left(0, v_i^n\right)$, where $v_i^n = \int_{(i-1)/n}^{i/n}\sigma_u^2 du$ and the following results hold:*

(a1)

$$E\left(S_n\right)=0 \text{ and } E\left(S_n^2\right)=1.$$

(a2)

$$\lim_{n\to\infty} E\left(S_nU_n\right)=a_1\sigma_{6,4},$$

where

$$a_1=\frac{1}{\sqrt{\tau}}\left(3\frac{k_{\frac{10}{3}}}{k_{\frac{4}{3}}}-2\frac{k_{\frac{7}{3}}^2}{k_1^2k_{\frac{4}{3}}^2}-2\frac{k_{\frac{7}{3}}}{k_1k_{\frac{4}{3}}}+1\right)\simeq -1.792629988661774.$$

(a3)

$$\lim_{n\to\infty}\sqrt{n}E\left(S_n^3\right)=a_2\sigma_{6,4},$$

where

$$a_2=\frac{1}{\tau^{3/2}}\left(k_6+3k_4-6\frac{(k_5+2k_3)}{k_1}+6\frac{(4-k_3^2)}{k_1^2}+12\frac{k_3}{k_1^3}+\frac{15}{k_1^4}-6\frac{k_3}{k_1^5}-\frac{k_3^2}{k_1^6}\right)\simeq 1.958608591285652.$$

(a4)

$$\lim_{n\to\infty}E\left(S_n^3U_n\right)=a_3\sigma_{6,4},$$

with

$$\begin{aligned} a_3 \;=\; & \frac{1}{\tau^{3/2}}\left(33+27\frac{k_{\frac{10}{3}}}{k_{\frac{4}{3}}}+\frac{3}{k_1^4}-\frac{18}{k_1^2}-30\frac{k_{\frac{7}{3}}^2}{k_1^2k_{\frac{4}{3}}^2}-30\frac{k_{\frac{7}{3}}}{k_1k_{\frac{4}{3}}}-12\frac{k_3}{k_1}\right.\\ & \left.-6\frac{k_{\frac{7}{3}}^2}{k_1^6k_{\frac{4}{3}}^2}-6\frac{k_{\frac{7}{3}}}{k_1^5k_{\frac{4}{3}}}-36\frac{k_3k_{\frac{10}{3}}}{k_1k_{\frac{4}{3}}}+9\frac{k_{\frac{10}{3}}}{k_1^4k_{\frac{4}{3}}}+18\frac{k_{\frac{10}{3}}}{k_1^2k_{\frac{4}{3}}}+24\frac{k_3k_{\frac{7}{3}}^2}{k_1^3k_{\frac{4}{3}}^2}+24\frac{k_3k_{\frac{7}{3}}}{k_1^2k_{\frac{4}{3}}}\right)\\ \simeq\; & 33.52851853541578. \end{aligned}$$

(a5)

$$\sqrt{n}E\left(S_n^2U_n\right)=O\left(1\right).$$

Remark 1 The bootstrap analogue of Lemma S2.4 replaces v_i^n with the local measure of volatility $\hat{v}_i^n$ and V_n with V_n^*, yielding for example

$$E\left(S_{n,1}^{*3}\right) = \frac{(k_6 - 3k_4 + 2)}{V_n^{*3/2}} n^{3/2} \sum_{i=1}^{n} \left(\hat{v}_i^n\right)^3.$$

Lemma S2.6 *Let X_t be described as in (10). Then, conditionally on the path of volatility, the following results hold:*

(a1)
$$E^*\left(S_n^*\right) = 0 \text{ and } E^*\left(S_n^{*2}\right) = 1.$$

(a2)
$$p\lim_{n\to\infty} E^*\left(S_n^* U_n^*\right) = a_1 \sigma_{6,4},$$

where a_1 is as in part (a2) of Lemma S2.5.

(a3)
$$p\lim_{n\to\infty}\left[\sqrt{n} E^*\left(S_n^{*3}\right)\right] = a_2 \sigma_{6,4},$$

where a_2 is as in part (a3) of Lemma S2.5.

(a4)
$$p\lim_{n\to\infty} E^*\left(S_n^{*3} U_n^*\right) = a_3 \sigma_{6,4},$$

where a_3 is as in part (a4) of Lemma S2.5.

(a5)
$$\sqrt{n} E^*\left(S_n^{*2} U_n^*\right) = O_P(1).$$

(a6) If in addition $n = O(k_n^2)$,
$$B_n^* = O_P(1).$$

S2.3 Proofs of auxiliary lemmas

Proof of Lemma S2.1. Let $\sigma_{i,n} \equiv (v_i^n)^{1/2}$. We first show that, for any $q_1, q_2 \geq 2$,

$$\sum_{i=1}^{n-1} (v_i^n)^{q_1} (v_{i+1}^n)^{q_2} - \sum_{i=1}^{n} (v_i^n)^{q_1+q_2} = O_P(n^{\frac{1}{2}-q_1-q_2}). \tag{S2.9}$$

We have:

$$\begin{aligned}
\left|\sum_{i=1}^{n-1} (v_i^n)^{q_1} (v_{i+1}^n)^{q_2} - \sum_{i=1}^{n} (v_i^n)^{q_1+q_2}\right| &= \left|\sum_{i=1}^{n-1} \sigma_{i,n}^{2q_1} \sigma_{i+1,n}^{2q_2} - \sum_{i=1}^{n} \sigma_{i,n}^{2(q_1+q_2)}\right| \\
&\leq \left|\sum_{i=1}^{n-1} \sigma_{i,n}^{2q_1}\left(\sigma_{i+1,n}^{2q_2} - \sigma_{i,n}^{2q_2}\right)\right| + \sigma_{n,n}^{2(q_1+q_2)} \\
&\leq \left(\sum_{i=1}^{n-1} \sigma_{i,n}^{4q_1}\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n-1} (\sigma_{i+1,n}^{2q_2} - \sigma_{i,n}^{2q_2})^2\right)^{\frac{1}{2}} + \sigma_{n,n}^{2(q_1+q_2)}.
\end{aligned}$$

The last inequality follows from Cauchy-Schwarz inequality. Thus, with $\psi_i = \sqrt{n}\sigma_{i,n}$, we have:

$$\left|\sum_{i=1}^{n-1}(v_i^n)^{q_1}(v_{i+1}^n)^{q_2} - \sum_{i=1}^{n}(v_i^n)^{q_1+q_2}\right| \leq n^{\frac{1}{2}-q_1-q_2}\left(\frac{1}{n}\sum_{i=1}^{n}\psi_i^{4q_1}\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n-1}\left(\psi_{i+1}^{2q_2}-\psi_i^{2q_2}\right)^2\right)^{\frac{1}{2}} + n^{-q_1-q_2}\psi_n^{2(q_1+q_2)}.$$

Note that following the same argument as Barndoff-Nielsen and Shephard's (2004) proof of their Eq. (14), we have: ψ_i's are uniformly bounded by $\sup_{1\leq s\leq t}\sigma(s) < \infty$ and $\sum_{i=1}^{n-1}\left(\psi_{i+1}^{2q_2}-\psi_i^{2q_2}\right)^2 = O_P(1)$. This establishes (S2.9).

To complete the proof, we have:

$$\begin{aligned}
&\left|\sum_{i=1}^{n-2}(v_i^n)^{q_1}(v_{i+1}^n)^{q_2}(v_{i+2}^n)^{q_3} - \sum_{i=1}^{n}(v_i^n)^{q_1+q_2+q_3}\right| \\
\leq\ &\left|\sum_{i=1}^{n-2}\sigma_{i,n}^{2q_1}\sigma_{i+1,n}^{2q_2}\left(\sigma_{i+2,n}^{2q_3}-\sigma_{i+1,n}^{2q_3}\right) + \left(\sum_{i=1}^{n-2}\sigma_{i,n}^{2q_1}\sigma_{i+1,n}^{2q_2+2q_3} - \sum_{i=1}^{n-1}\sigma_{i,n}^{2(q_1+q_2+q_3)}\right)\right| + \sigma_{n,n}^{2(q_1+q_2+q_3)} \\
\equiv\ &|a_n+b_n| + \sigma_{n,n}^{2(q_1+q_2+q_3)} = |a_n+b_n| + O_P(n^{-q_1-q_2-q_3}).
\end{aligned}$$

From (S2.9), we can claim that $b_n = O_P(n^{\frac{1}{2}-q_1-q_2-q_3})$. It remains to show that $a_n = O_P(n^{\frac{1}{2}-q_1-q_2-q_3})$. By the Cauchy-Schwarz inequality, we have:

$$\begin{aligned}
|a_n| &\leq \left(\sum_{i=1}^{n-2}\sigma_{i,n}^{4q_1}\sigma_{i+1,n}^{4q_2}\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n-2}\left(\sigma_{i+2,n}^{2q_3}-\sigma_{i+1,n}^{2q_3}\right)^2\right)^{\frac{1}{2}} \\
&\leq n^{\frac{1}{2}-q_1-q_2-q_3}\left(\frac{1}{n}\sum_{i=1}^{n-1}\psi_i^{4q_1}\psi_{i+1}^{4q_2}\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n-1}\left(\psi_{i+1}^{2q_3}-\psi_i^{2q_3}\right)^2\right)^{\frac{1}{2}}.
\end{aligned}$$

By the same arguments as previously, we conclude that $a_n = O_P(n^{\frac{1}{2}-q_1-q_2-q_3})$, which concludes the proof.

Proof of Lemma S2.2. Write:

$$n^{-1+q/2}\sum_{i=K}^{n}\prod_{k=1}^{K}\left(v_{i-k+1}^n\right)^{q_k/2} - \overline{\sigma^q} = n^{-1+q/2}\left[\sum_{i=K}^{n}\prod_{k=1}^{K}\left(v_{i-k+1}^n\right)^{q_k/2} - \sum_{i=1}^{n}(v_i^n)^{q/2}\right] + \left[n^{-1+q/2}\sum_{i=1}^{n}(v_i^n)^{q/2} - \overline{\sigma^q}\right].$$

From Lemma S2.1, the first term in the RHS is $o_P(1)$ and by Riemann integrability of σ_t, the second term is $o_P(1)$ (see Barndorff-Nielsen and Shephard (2004, p.10).

Proof of Lemma S2.3. We use a similar expansion to that of Eq. (13) of Barndoff-Nielsen and Shephard (2004). Let $\sigma_{i,n} = (v_i^n)^{1/2}$. Then, $\Xi_n \equiv \sum_{i=1}^n \sigma_{i,n}^2 - \sum_{i=2}^n \sigma_{i,n}\sigma_{i-1,n}$ is equal to

$$\sum_{i=2}^{n}\sigma_{i,n}(\sigma_{i,n}-\sigma_{i-1,n}) + \sigma_{1,n}^2 = \sum_{i=2}^{n}\frac{\sigma_{i,n}}{\sigma_{i,n}+\sigma_{i-1,n}}(\sigma_{i,n}^2-\sigma_{i-1,n}^2) + \sigma_{1,n}^2.$$

Alternatively, Ξ_n can also be written as

$$\sum_{i=2}^{n}\sigma_{i-1,n}^2 - \sum_{i=2}^{n}\sigma_{i,n}\sigma_{i-1,n} + \sigma_{n,n}^2 = \sum_{i=2}^{n}\frac{\sigma_{i-1,n}}{\sigma_{i,n}+\sigma_{i-1,n}}(\sigma_{i-1,n}^2-\sigma_{i,n}^2) + \sigma_{n,n}^2.$$

It results that

$$\Xi_n = \frac{1}{2}\sum_{i=2}^{n}\frac{\sigma_{i,n}-\sigma_{i-1,n}}{\sigma_{i,n}+\sigma_{i-1,n}}(\sigma_{i,n}^2-\sigma_{i-1,n}^2) + \frac{1}{2}(\sigma_{1,n}^2+\sigma_{n,n}^2) = \frac{1}{2}\sum_{i=2}^{n}\frac{(\sigma_{i,n}^2-\sigma_{i-1,n}^2)^2}{(\sigma_{i,n}+\sigma_{i-1,n})^2} + \frac{1}{2}(\sigma_{1,n}^2+\sigma_{n,n}^2) \equiv C_n + D_n.$$

We show that $nC_n \xrightarrow{p} 0$ and $nD_n \xrightarrow{p} \frac{1}{2}(\sigma_0^2+\sigma_1^2)$ as $n\to\infty$. Since σ_u^2 is bounded on $[0,1]$ and away from 0, we have: $\underline{\sigma}^2 = \inf_{u\in[0,1]}\sigma_u^2 > 0$ and, for all $i=1,\ldots,n$, $\sigma_{i,n}^2 \geq \frac{\underline{\sigma}^2}{n} > 0$. Thus,

$$C_n \leq \frac{n}{8\underline{\sigma}^2}\sum_{i=2}^{n}(\sigma_{i,n}^2 - \sigma_{i-1,n})^2.$$

Also, by pathwise continuity of σ_u^2, there exists $\xi_i \in \left[\frac{i-1}{n}, \frac{i}{n}\right]$ such that $\sigma_{i,n}^2 \equiv \int_{\frac{i-1}{n}}^{\frac{i}{n}} \sigma_u^2 du = \frac{\sigma_{\xi_i}^2}{n}$. Hence,

$$nC_n \leq \frac{1}{8\underline{\sigma}^2}\sum_{i=2}^{n}(\sigma_{\xi_i}^2 - \sigma_{\xi_{i-1}}^2)^2.$$

The $L^2(P)$-Hölder continuity of σ_u^2 implies that, for some $K>0$, and for all $i=1,\ldots,n$,

$$E\left((\sigma_{\xi_i}^2 - \sigma_{\xi_{i-1}}^2)^2\right) \leq K\frac{2^{2\delta}}{n^{2\delta}}.$$

It follows that

$$E\left(\sum_{i=2}^{n}(\sigma_{\xi_i}^2 - \sigma_{\xi_{i-1}}^2)^2\right) \leq K2^{2\delta}\frac{n-1}{n^{2\delta}} \to 0,$$

as $n\to\infty$. We conclude by the Markov inequality that $\sum_{i=2}^{n}(\sigma_{\xi_i}^2 - \sigma_{\xi_{i-1}}^2)^2 = o_P(1)$. It follows that $nC_n = o_P(1)$ since $1/\underline{\sigma}^2 = O_P(1)$.

Next, using the fact that $\sigma_{1,n}^2 = \frac{\sigma_{\xi_1}^2}{n}$ with $\xi_1 \in \left[0, \frac{1}{n}\right]$, we deduce from the right-continuity of σ_u^2 at $u=0$ that $n\sigma_{1,n}^2 \xrightarrow{p} \sigma_0^2$. We obtain along the same line that $n\sigma_{n,n}^2 \xrightarrow{p} \sigma_1^2$ using left continuity at $u=1$ establishing that $nD_n \xrightarrow{p} \frac{1}{2}(\sigma_0^2+\sigma_1^2)$.

Proof of Lemma S2.4. In the following recall that $k_2=1$, $k_4=3$, and $k_6=15$. Let

$$K_{1n} = \frac{\sqrt{n}}{\sqrt{V_n}}, \quad K_{2n} = \frac{\sqrt{n}}{k_1^2\sqrt{V_n}}, \quad \text{and} \quad K_{3n} = \frac{\tau n^{3/2}}{k_{\frac{4}{3}}^3 V_n}.$$

Write

$$\begin{aligned}
S_{n,1} &= K_{1n}\sum_{i=1}^{n}\left(r_i^2 - E\left(r_i^2\right)\right) \equiv K_{1n}\sum_{i=1}^{n} a_i,\\
S_{n,2} &= K_{2n}\sum_{i=2}^{n}\left(|r_i r_{i-1}| - E\left(|r_i r_{i-1}|\right)\right) \equiv K_{2n}\sum_{i=1}^{n} b_{i,i-1},\\
U_n &= K_{3n}\sum_{i=3}^{n}\left(|r_i r_{i-1} r_{i-2}|^{4/3} - E\left(|r_i r_{i-1} r_{i-2}|^{4/3}\right)\right) \equiv K_{3n}\sum_{i=1}^{n} c_{i,i-1,i-2}.
\end{aligned}$$

(a1) Follows directly given the definition of $S_{n,1}$ and $S_{n,2}$.

(a2)

$$E\left(S_{n,1}U_n\right) = \frac{\tau n^2}{k_{\frac{4}{3}}^3 V_n^{3/2}}\sum_{i=1}^{n}\sum_{j=3}^{n} I_{i,j}$$

where

$$\begin{aligned}
I_{i,j} &= E\left(a_i c_{j,j-1,j-2}\right)\\
&= E\left[\left(r_i^2 - v_i^n\right)\left(|r_{j-2}|^{4/3}|r_{j-1}|^{4/3}|r_j|^{4/3} - k_{\frac{4}{3}}^3\left(v_{j-2}^n\right)^{2/3}\left(v_{j-1}^n\right)^{2/3}\left(v_j^n\right)^{2/3}\right)\right].
\end{aligned}$$

The non zero contribution to $E\left(S_{n,1}U_n\right)$ are when $i=j$; $i=j-2$ and $i=j-1$. In particular, we have

$$\begin{aligned}
\sum_{i=j} I_{i,j} &= \sum_{i=3}^{n} E\left(a_i c_{i,i-1,i-2}\right) \\
&= \sum_{i=3}^{n} E\left(\left|r_{i-2}\right|^{4/3}\left|r_{i-1}\right|^{4/3}\left|r_i\right|^{10/3} - k_{\frac{4}{3}}^3 \left(v_{i-2}^n\right)^{2/3}\left(v_{i-1}^n\right)^{2/3}\left(v_i^n\right)^{2/3} r_i^{*2}\right) \\
&= \left(k_{\frac{4}{3}}^2 k_{\frac{10}{3}} - k_{\frac{4}{3}}^3\right)\sum_{i=3}^{n}\left(v_{i-2}^n\right)^{2/3}\left(v_{i-1}^n\right)^{2/3}\left(v_i^n\right)^{5/3},
\end{aligned}$$

$$\begin{aligned}
\sum_{i=j-2} I_{i,j} &= \sum_{i=1}^{n-2} E\left[r_i^2\left(\left|r_i\right|^{4/3}\left|r_{i+1}\right|^{4/3}\left|r_{i+2}\right|^{4/3} - k_{\frac{4}{3}}^3\left(v_i^n\right)^{2/3}\left(v_{i+1}^n\right)^{2/3}\left(v_{i+2}^n\right)^{2/3}\right)\right] \\
&= \left(k_{\frac{4}{3}}^2 k_{\frac{10}{3}} - k_{\frac{4}{3}}^3\right)\sum_{i=1}^{n-2}\left(v_i^n\right)^{5/3}\left(v_{i+1}^n\right)^{2/3}\left(v_{i+2}^n\right)^{2/3},
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=j-1} I_{i,j} &= \sum_{i=2}^{n-1} E\left[r_i^2\left(\left|r_{i-1}\right|^{4/3}\left|r_i\right|^{4/3}\left|r_{i+1}\right|^{4/3} - k_{\frac{4}{3}}^3\left(v_{i-1}^n\right)^{2/3}\left(v_i^n\right)^{2/3}\left(v_{i+1}^n\right)^{2/3}\right)\right] \\
&= \left(k_{\frac{4}{3}}^2 k_{\frac{10}{3}} - k_{\frac{4}{3}}^3\right)\sum_{i=2}^{n-1}\left(v_{i-1}^n\right)^{2/3}\left(v_i^n\right)^{5/3}\left(v_{i+1}^n\right)^{2/3}.
\end{aligned}$$

Therefore,

$$E\left(S_{n,1}U_n\right) = \frac{\tau\left(k_{\frac{4}{3}}^2 k_{\frac{10}{3}} - k_{\frac{4}{3}}^3\right)}{k_{\frac{4}{3}}^3 V_n^{3/2}} n^2 \left[\begin{array}{l} \sum_{i=3}^n \left(v_{i-2}^n\right)^{2/3}\left(v_{i-1}^n\right)^{2/3}\left(v_i^n\right)^{5/3} \\ +\sum_{i=3}^n \left(v_{i-2}^n\right)^{5/3}\left(v_{i-1}^n\right)^{2/3}\left(v_i^n\right)^{2/3} \\ +\sum_{i=3}^n \left(v_{i-2}^n\right)^{2/3}\left(v_{i-1}^n\right)^{5/3}\left(v_i^n\right)^{2/3} \end{array}\right].$$

(a3)

$$E\left(S_{n,2}U_n\right) = \frac{\tau n^2}{k_1^2 k_{\frac{4}{3}}^3 V_n^{3/2}} \sum_{i=2}^{n}\sum_{j=3}^{n} I_{i,j}$$

where

$$\begin{aligned}
I_{i,j} &= E\left(b_{i,i-1} c_{j,j-1,j-2}\right) \\
&= E\left[\left(\left|r_{i-1}\right|\left|r_i\right| - k_1^2\sqrt{v_{i-1}^n v_i^n}\right)\left(\left|r_{j-2}\right|^{4/3}\left|r_{j-1}\right|^{4/3}\left|r_j\right|^{4/3} - k_{\frac{4}{3}}^3\left(v_{j-2}^n\right)^{2/3}\left(v_{j-1}^n\right)^{2/3}\left(v_j^n\right)^{2/3}\right)\right].
\end{aligned}$$

The non zero contributions to $E\left(S_{n,2}U_n\right)$ are when $i=j$; $i=j-1, i=j-2$; and $i=j+1$. In particular, we have

$$\begin{aligned}
\sum_{i=j} I_{i,j} &= \sum_{i=3}^{n} E\left[\left|r_{i-1}\right|\left|r_i\right|\left(\left|r_{i-2}\right|^{4/3}\left|r_{i-1}\right|^{4/3}\left|r_i\right|^{4/3} - k_{\frac{4}{3}}^3\left(v_{i-2}^n\right)^{2/3}\left(v_{i-1}^n\right)^{2/3}\left(v_i^n\right)^{2/3}\right)\right] \\
&= \sum_{i=3}^{n} E\left(\left|r_{i-2}\right|^{4/3}\left|r_{i-1}\right|^{7/3}\left|r_i\right|^{7/3} - k_{\frac{4}{3}}^3\left(v_{i-2}^n\right)^{2/3}\left(v_{i-1}^n\right)^{2/3}\left(v_i^n\right)^{2/3}\left|r_{i-1}\right|\left|r_i\right|\right) \\
&= \left(k_{\frac{4}{3}} k_{\frac{7}{3}}^2 - k_1^2 k_{\frac{4}{3}}^3\right)\sum_{i=3}^{n}\left(v_{i-2}^n\right)^{2/3}\left(v_{i-1}^n\right)^{7/6}\left(v_i^n\right)^{7/6},
\end{aligned}$$

$$\sum_{i=j-1} I_{i,j} = \sum_{i=2}^{n-1} E\left[|r_{i-1}|\,|r_i|\left(|r_{i-1}|^{4/3}\,|r_i|^{4/3}\,|r_{i+1}|^{4/3} - k_{\frac{4}{3}}^3 \left(v_{i-1}^n\right)^{2/3} \left(v_i^n\right)^{2/3} \left(v_{i+1}^n\right)^{2/3}\right)\right]$$
$$= \left(k_{\frac{4}{3}} k_{\frac{7}{3}}^2 - k_1^2 k_{\frac{4}{3}}^3\right) \sum_{i=3}^{n} \left(v_{i-2}^n\right)^{7/6} \left(v_{i-1}^n\right)^{7/6} \left(v_i^n\right)^{2/3},$$

$$\sum_{i=j-2} I_{i,j} = \sum_{i=2}^{n-2} E\left[|r_{i-1}|\,|r_i|\left(|r_i|^{4/3}\,|r_{i+1}|^{4/3}\,|r_{i+2}|^{4/3} - k_{\frac{4}{3}}^3 \left(v_i^n\right)^{2/3} \left(v_{i+1}^n\right)^{2/3} \left(v_{i+2}^n\right)^{2/3}\right)\right]$$
$$= \left(k_1 k_{\frac{4}{3}}^2 k_{\frac{7}{3}} - k_1^2 k_{\frac{4}{3}}^3\right) \sum_{i=4}^{n} \left(v_{i-3}^n\right)^{1/2} \left(v_{i-2}^n\right)^{7/6} \left(v_{i-1}^n\right)^{2/3} \left(v_i^n\right)^{2/3},$$

and

$$\sum_{i=j+1} I_{i,j} = \sum_{i=4}^{n} E\left[|r_{i-1}|\,|r_i|\left(|r_{i-3}|^{4/3}\,|r_{i-2}|^{4/3}\,|r_{i-1}|^{4/3} - k_{\frac{4}{3}}^3 \left(v_{i-3}^n\right)^{2/3} \left(v_{i-2}^n\right)^{2/3} \left(v_{i-1}^n\right)^{2/3}\right)\right]$$
$$= \left(k_1 k_{\frac{4}{3}}^2 k_{\frac{7}{3}} - k_1^2 k_{\frac{4}{3}}^3\right) \sum_{i=4}^{n} \left(v_{i-3}^n\right)^{2/3} \left(v_{i-2}^n\right)^{2/3} \left(v_{i-1}^n\right)^{7/6} \left(v_i^n\right)^{1/2}.$$

It follows that

$$E\left(S_{n,2}U_n\right) = \frac{\tau\left(k_{\frac{4}{3}} k_{\frac{7}{3}}^2 - k_1^2 k_{\frac{4}{3}}^3\right)}{k_1^2 k_{\frac{4}{3}}^3 V_n^{3/2}} n^2 \left[\begin{array}{c} \sum_{i=3}^n \left(v_{i-2}^n\right)^{2/3} \left(v_{i-1}^n\right)^{7/6} \left(v_i^n\right)^{7/6} \\ +\sum_{i=3}^n \left(v_{i-2}^n\right)^{7/6} \left(v_{i-1}^n\right)^{7/6} \left(v_i^n\right)^{2/3} \end{array}\right]$$
$$+\frac{\tau\left(k_1 k_{\frac{4}{3}}^2 k_{\frac{7}{3}} - k_1^2 k_{\frac{4}{3}}^3\right)}{k_1^2 k_{\frac{4}{3}}^3 V_n^{3/2}} n^2 \left[\begin{array}{c} \sum_{i=4}^n \left(v_{i-3}^n\right)^{1/2} \left(v_{i-2}^n\right)^{7/6} \left(v_{i-1}^n\right)^{2/3} \left(v_i^n\right)^{2/3} \\ +\sum_{i=4}^n \left(v_{i-3}^n\right)^{2/3} \left(v_{i-2}^n\right)^{2/3} \left(v_{i-1}^n\right)^{7/6} \left(v_i^n\right)^{1/2} \end{array}\right].$$

(a4) We have

$$E\left(S_{n,1}^2 U_n\right) = \frac{\tau n^{5/2}}{k_{\frac{4}{3}}^3 V_n^2} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=2}^n I_{i,j,k},$$

where

$$I_{i,j,k} = E\left[\left(r_i^2 - v_i^n\right)\left(r_j^2 - v_j^n\right)\left(|r_{k-2}|^{4/3}\,|r_{k-1}|^{4/3}\,|r_k|^{4/3} - k_{\frac{4}{3}}^3 \left(v_{k-2}^n\right)^{2/3} \left(v_{k-1}^n\right)^{2/3} \left(v_k^n\right)^{2/3}\right)\right].$$

The non zero contributions to $E\left(S_{n,1}^2 U_n\right)$ are from triplets (i,j,k) in

$$\{(k-2,k-2,k),(k-2,k-1,k),(k-2,k,k),(k-1,k-2,k),(k-1,k-1,k),$$
$$(k-1,k,k),(k,k-2,k),(k,k-1,k),(k,k,k) : k=1,\ldots,n\}$$

with the convention that out of range terms are set to 0. Tedious but straightforward calculations show that the sum of $I_{i,j,k}$ of each relevant triplet is of order $O_P(n^{-3})$, by Lemma 3.2, completing the proof.

(a5) We have

$$E\left(S_{n,1}S_{n,2}U_n\right) = \frac{\tau n^{5/2}}{k_1^2 k_{\frac{4}{3}}^3 V_n^2} \sum_{i=1}^n \sum_{j=2}^n \sum_{k=3}^n I_{i,j,k},$$

where

$$I_{i,j,k} = E\left[\left(r_i^2 - v_i^n\right)\left(|r_{j-1}|\,|r_j| - k_1^2\left(v_{j-1}^n\right)^{1/2}\left(v_j^n\right)^{1/2}\right)\left(\begin{array}{c} |r_{k-2}|^{4/3}\,|r_{k-1}|^{4/3}\,|r_k|^{4/3} \\ -k_{\frac{4}{3}}^3\left(v_{k-2}^n\right)^{2/3}\left(v_{k-1}^n\right)^{2/3}\left(v_k^n\right)^{2/3} \end{array}\right)\right].$$

The non zero contributions to $E\left(S_{n,1}S_{n,2}U_n\right)$ are from the triplets (i,j,k) in:

$$\begin{array}{c}\{(k-3,k-2,k),(k-2,k-2,k),(k-1,k-2,k),(k,k-2,k),(k-2,k-1,k),\\ (k-1,k-1,k),(k,k-1,k),(k-2,k,k),(k-1,k,k),(k,k,k),\\ (k-2,k+1,k),(k-1,k+1,k),(k,k+1,k),(k+1,k+1,k) : k=1,\ldots,n\}\end{array}$$

with the convention that out of range terms are set to 0. Tedious but straightforward calculations show that the sum of $I_{i,j,k}$ of each relevant triplet, using Lemma 3.2, is of order $O_P(n^{-3})$ yielding the expected result.

(a6) We have

$$E\left(S_{n,2}^2 U_n\right) = \frac{\tau n^{5/2}}{k_1^4 k_{\frac{4}{3}}^3 V_n^2}\sum_{i=1}^n\sum_{j=2}^n\sum_{k=3}^n I_{i,j,k},$$

where

$$I_{i,j,k} = E\left[\left(\begin{array}{c} |r_{i-1}|\,|r_i| \\ -k_1^2\left(v_{i-1}^n\right)^{1/2}\left(v_i^n\right)^{1/2}\end{array}\right)\left(\begin{array}{c} |r_{j-1}|\,|r_j| \\ -k_1^2\left(v_{j-1}^n\right)^{1/2}\left(v_j^n\right)^{1/2}\end{array}\right)\left(\begin{array}{c} |r_{k-2}|^{4/3}\,|r_{k-1}|^{4/3}\,|r_k|^{4/3} \\ -k_{\frac{4}{3}}^3\left(v_{k-2}^n\right)^{2/3}\left(v_{k-1}^n\right)^{2/3}\left(v_k^n\right)^{2/3} \end{array}\right)\right].$$

The non zero contribution to $E\left(S_{n,2}^2 U_n\right)$ are from the triplets (i,j,k) in:

$$\begin{array}{c}\{(k-3,k-2,k),(k-2,k-3,k),(k-2,k-2,k),(k-2,k-1,k),(k-2,k,k),\\ (k-2,k+1,k),(k-1,k-2,k),(k-1,k-1,k),(k-1,k,k),(k-1,k+1,k),\\ (k,k-2,k),(k,k-1,k),(k,k,k),(k,k+1,k),(k+1,k-2,k),(k+1,k-1,k),\\ (k+1,k,k),(k+1,k+1,k),(k+1,k+2,k),(k+2,k+1,k) : k=1,\ldots,n\},\end{array}$$

once again, with the convention that out of range terms are set to 0. Tedious but straightforward calculations show that the sum of $I_{i,j,k}$ over each relevant triplet, using Lemma 3.2, is of order $O_P(n^{-3})$, yielding the expected result.

(a7)

$$E\left(S_{n,1}^3\right) = \frac{n^{3/2}}{V_n^{3/2}}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n E\left[\left(r_i^2 - v_i^n\right)\left(r_j^2 - v_j^n\right)\left(r_k^2 - v_k^n\right)\right].$$

The only non zero contribution to $E\left(S_{n,1}^3\right)$ is when $i=j=k$. Then we have

$$\begin{aligned} E\left(S_{n,1}^3\right) &= \frac{n^{3/2}}{V_n^{*3/2}}\sum_{i=1}^n E\left(r_i^2 - v_i^n\right)^3 \\ &= \frac{n^{3/2}}{V_n^{3/2}}\sum_{i=1}^n E\left(r_i^6 - 3v_i^n r_i^4 + 3\left(v_i^n\right)^2 r_i^2 - \left(v_i^n\right)^3\right) \\ &= \frac{(k_6 - 3k_4 + 2)}{V_n^{3/2}} n^{3/2}\sum_{i=1}^n \left(v_i^n\right)^3 \end{aligned}$$

(a8)

$$E\left(S_{n,1}^2 S_{n,2}\right) = \frac{n^{3/2}}{k_1^2 V_n^{3/2}} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=2}^n I_{i,j,k},$$

where

$$I_{i,j,k} = E\left[\left(r_i^2 - v_i^n\right)\left(r_j^2 - v_j^n\right)\left(|r_{k-1}|\,|r_k| - k_1^2\sqrt{v_{k-1}^n v_k^n}\right)\right].$$

The non zero contribution to $E\left(S_{n,1}^2 S_{n,2}\right)$ are when $i = j = k$; $i = j, k = i+1$; $i = k, j = i-1$ and $i = k-1, j = k$. In particular, we have

$$\begin{aligned}
\sum_{i=j=k} I_{i,j,k} &= \sum_{i=2}^n E\left[\left(r_i^2 - v_i^n\right)^2\left(|r_{i-1}|\,|r_i| - k_1^2\sqrt{v_{i-1}^n v_i^n}\right)\right] \\
&= \sum_{i=2}^n E\left[\left(r_i^4 - 2v_i^n r_i^2\right)\left(|r_{i-1}|\,|r_i| - k_1^2\sqrt{v_{i-1}^n v_i^n}\right)\right] \\
&= \sum_{i=2}^n E\left[\left(|r_{i-1}|\,|r_i|^5 - k_1^2\sqrt{v_{i-1}^n v_i^n}\,|r_i|^4 - 2\hat{v}_i^n\,|r_{i-1}|\,|r_i|^3 + 2k_1^2\left(v_{i-1}^n\right)^{1/2}\left(v_i^n\right)^{3/2} r_i^2\right)\right] \\
&= \left(k_1 k_5 - k_1^2 k_4 - 2k_1 k_3 + 2k_1^2\right)\sum_{i=2}^n \left(v_{i-1}^n\right)^{1/2}\left(v_i^n\right)^{5/2},
\end{aligned}$$

$$\begin{aligned}
\sum_{i=j,k=i+1} I_{i,j,k} &= \sum_{i=1}^{n-1} E\left[\left(r_i^2 - v_i^n\right)^2\left(|r_i|\,|r_{i+1}| - k_1^2\sqrt{v_i^n v_{i+1}^n}\right)\right] \\
&= \sum_{i=1}^{n-1} E\left[\left(r_i^4 - 2v_i^n r_i^2\right)\left(|r_i|\,|r_{i+1}| - k_1^2\sqrt{v_i^n v_{i+1}^n}\right)\right] \\
&= \left(k_1 k_5 - k_1^2 k_4 - 2k_1 k_3 + 2k_1^2\right)\sum_{i=1}^{n-1} \left(v_i^n\right)^{5/2}\left(v_{i+1}^n\right)^{1/2},
\end{aligned}$$

$$\begin{aligned}
\sum_{i=k,j=i-1} I_{i,j,k} &= \sum_{i=2}^n E\left[\left(r_i^2 - v_i^n\right)\left(r_{i-1}^2 - v_{i-1}^n\right)\left(|r_{i-1}|\,|r_i| - k_1^2\sqrt{v_{i-1}^n v_i^n}\right)\right] \\
&= \sum_{i=2}^n E\left[\left(r_{i-1}^2 r_i^2 - v_{i-1}^n r_i^2 - v_i^n r_{i-1}^2\right)\left(|r_{i-1}|\,|r_i| - k_1^2\sqrt{v_{i-1}^n v_i^n}\right)\right] \\
&= \left(k_3^2 - 2k_1 k_3 + k_1^2\right)\sum_{i=2}^n \left(v_{i-1}^n\right)^{3/2}\left(v_i^n\right)^{3/2},
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=k-1,j=k} I_{i,j,k} &= \sum_{i=1}^{n-1} E\left[\left(r_i^2 - v_i^n\right)\left(r_{i+1}^2 - v_{i+1}^n\right)\left(|r_i|\,|r_{i+1}| - k_1^2\sqrt{v_i^n v_{i+1}^n}\right)\right] \\
&= \left(k_3^2 - 2k_1 k_3 + k_1^2\right)\sum_{i=1}^{n-1} \left(v_i^n\right)^{3/2}\left(v_{i+1}^n\right)^{3/2}.
\end{aligned}$$

Thus

$$\begin{aligned}E\left(S_{n,1}^2 S_{n,2}\right) &= \frac{\left(k_1k_5 - k_1^2k_4 - 2k_1k_3 + 2k_1^2\right)}{k_1^2 V_n^{3/2}} n^{3/2} \left[\sum_{i=2}^{n} \left(v_{i-1}^n\right)^{1/2} \left(v_i^n\right)^{5/2} + \sum_{i=2}^{n} \left(v_{i-1}^n\right)^{5/2} \left(v_i^n\right)^{1/2}\right] \\ &+2\frac{\left(k_3^2 - 2k_1k_3 + k_1^2\right)}{k_1^2 V_n^{*3/2}} n^{3/2} \sum_{i=2}^{n} \left(v_{i-1}^n\right)^{3/2} \left(v_i^n\right)^{3/2}.\end{aligned}$$

(a9)

$$E\left(S_{n,1} S_{n,2}^2\right) = \frac{n^{3/2}}{k_1^4 V_n^{3/2}} \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} I_{i,j,k},$$

where

$$I_{i,j,k} = E\left[\left(r_i^2 - v_i^n\right)\left(|r_{j-1}|\,|r_j| - k_1^2\sqrt{v_{j-1}^n v_j^n}\right)\left(|r_{k-1}|\,|r_k| - k_1^2\sqrt{v_{k-1}^n v_k^n}\right)\right].$$

The non zero contribution to $E\left(S_{n,1}S_{n,2}^2\right)$ are from the triplets (i,j,k) in:

$$\{(k-2,k-1,k),(k-1,k-1,k),(k,k-1,k),(k-1,k,k),(k,k,k),\\ (k-1,k+1,k),(k,k+1,k),(k+1,k+1,k) : k=1,\dots,n\}.$$

Some tedious but straightforward calculations yield:

$$\begin{aligned}E\left(S_{n,1}S_{n,2}^2\right) &= \frac{2\left(1-k_1^3k_3+k_1^4\right)}{k_1^4V_n^{3/2}} n^{3/2}\sum_{i=2}^{n}\left[\left(v_{i-1}^n\right)\left(v_i^n\right)^2 + \left(v_{i-1}^n\right)^2\left(v_i^n\right)\right] \\ &+\frac{2\left(k_1^2-k_1^3k_3+k_1^4\right)}{k_1^4V_n^{3/2}} n^{3/2}\sum_{i}\left[\left(v_{i-2}^n\right)^{1/2}\left(v_{i-1}^n\right)^2\left(v_i^n\right)^{1/2} + \left(v_i^n\right)^2\left(v_{i-1}^n\right)^{1/2}\left(v_{i+1}^n\right)^{1/2}\right] \\ &+\frac{\left(k_1k_3-k_1^3k_3-k_1^2+k_1^4\right)}{k_1^4V_n^{3/2}} n^{3/2}\sum_{i}\left[\left(v_{i-2}^n\right)^{1/2}\left(v_{i-1}^n\right)\left(v_i^n\right)^{3/2} + \left(v_{i-2}^n\right)^{3/2}\left(v_{i-1}^n\right)\left(v_i^n\right)^{1/2}\right] \\ &+\frac{\left(k_1^4-k_1^2-k_1^3k_3+k_1k_3\right)}{k_1^4V_n^{3/2}} n^{3/2}\sum_{i}\left[\left(v_{i-1}^n\right)^{3/2}\left(v_i^n\right)\left(v_{i+1}^n\right)^{1/2} + \left(v_i^n\right)\left(v_{i-1}^n\right)^{1/2}\left(v_{i+1}^n\right)^{3/2}\right].\end{aligned}$$

(a10)

$$E\left(S_{n,2}^3\right) = \frac{n^{3/2}}{k_1^6 V_n^{3/2}} \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} I_{i,j,k},$$

where

$$I_{i,j,k} = E\left[\left(|r_{i-1}|\,|r_i| - k_1^2\sqrt{v_{i-1}^n v_i^n}\right)\left(|r_{j-1}|\,|r_j| - k_1^2\sqrt{v_{j-1}^n v_j^n}\right)\left(|r_{k-1}|\,|r_k| - k_1^2\sqrt{v_{k-1}^n v_k^n}\right)\right].$$

The only non zero contribution to $E\left(S_{n,2}^3\right)$ are from the triplets (i,j,k) in:

$$\{(k-2,k-1,k),(k-1,k-2,k),(k-1,k-1,k),(k-1,k,k),(k-1,k+1,k),\\ (k,k-1,k),(k,k,k),(k,k+1,k),(k+1,k-1,k),(k+1,k,k),\\ (k+1,k+1,k),(k+1,k+2,k),(k+2,k+1,k) : k=1,\dots,n\}.$$

Some tedious but straightforward calculations yield:

$$\begin{aligned}
E\left(S_{n,2}^3\right) &= \frac{\left(k_3^2-3k_1^2+2k_1^6\right)}{k_1^6 V_n^{3/2}} n^{3/2} \sum_{i=2}^{n} \left(v_{i-1}^n\right)^{3/2} \left(v_i^n\right)^{3/2} \\
&+ \frac{2\left(k_1k_3-k_1^2-2k_1^4+2k_1^6\right)}{k_1^6 V_n^{3/2}} n^{3/2} \sum_{i=3}^{n} \left[\left(v_{i-2}^n\right)^{1/2}\left(v_{i-1}^n\right)^{3/2} v_i^n + v_{i-2}^n \left(v_{i-1}^n\right)^{3/2}\left(v_i^n\right)^{1/2}\right] \\
&+ \frac{\left(2k_1^6-2k_1^4-k_1^2+k_1k_3\right)}{k_1^6 V_n^{3/2}} n^{3/2} \sum_{i} \left[\left(v_{i-2}^n\right)\left(v_{i-1}^n\right)^{3/2}\left(v_i^n\right)^{1/2} + \left(v_{i-1}^n\right)^{1/2}\left(v_i^n\right)^{3/2}\left(v_{i+1}^n\right)\right] \\
&+ \frac{6\left(k_1^6-2k_1^4+k_1^2\right)}{k_1^6 V_n^{3/2}} n^{3/2} \sum_{i} \left(v_{i-2}^n\right)^{1/2}\left(v_{i-1}^n\right)\left(v_i^n\right)\left(v_{i+1}^n\right)^{1/2}.
\end{aligned}$$

(a11)

$$E\left(S_{n,1}^3 U_n\right) = \frac{\tau n^3}{k_{\frac{4}{3}}^3 V_n^{5/2}} \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=3}^{n} I_{i,j,k,l}.$$

where

$$I_{i,j,k,l} = E\left[\left(r_i^2-v_i^n\right)\left(r_j^2-v_j^n\right)\left(r_k^2-v_k^n\right)\left(|r_{l-2}|^{4/3}|r_{l-1}|^{4/3}|r_l|^{4/3} - k_{\frac{4}{3}}^3 \left(v_{l-2}^n\right)^{2/3}\left(v_{l-1}^n\right)^{2/3}\left(v_l^n\right)^{2/3}\right)\right].$$

The non zero contribution to $E\left(S_{n,1}^3 U_n\right)$ is given as follows

$$E\left(S_{n,1}^3 U_n\right) = \frac{\tau n^3}{k_{\frac{4}{3}}^3 V_n^{5/2}} \left[3\left(\sum_i E(a_i^2)\right)\left(\sum_i E(a_i+a_{i-1}+a_{i-2})c_{i,i-1,i-2}\right) + O(n^{-4})\right].$$

Hence, we have

$$\begin{aligned}
E\left(S_{n,1}^3 U_n\right) &= \frac{\tau n^3}{k_{\frac{4}{3}}^3 V_n^{5/2}} \left[\begin{array}{c} 3\left((k_4-k_2)\left(\sum_{i=1}^n (v_i^n)^2\right)\right)\left(k_{\frac{4}{3}}^2 k_{\frac{10}{3}} - k_{\frac{4}{3}}^3\right) \\ \times \left[\begin{array}{c} \sum_{i=3}^n \left(v_{i-2}^n\right)^{2/3}\left(v_{i-1}^n\right)^{2/3}\left(v_i^n\right)^{5/3} \\ +\sum_{i=3}^n \left(v_{i-2}^n\right)^{5/3}\left(v_{i-1}^n\right)^{2/3}\left(v_i^n\right)^{2/3} \\ +\sum_{i=3}^n \left(v_{i-2}^n\right)^{2/3}\left(v_{i-1}^n\right)^{5/3}\left(v_i^n\right)^{2/3} \end{array}\right] \end{array}\right] \\
&\quad + O(n^{-1}).
\end{aligned}$$

(a12) We can write

$$\begin{aligned}
& E\left(S_{n,1}^2 S_{n,2} U_n\right) \\
=\ & K_{1n}^2 K_{2n} K_{3n} E\left(\left[\sum_i a_i\right]^2 \left[\sum_i b_{i,i-1}\right]\left[\sum_i c_{i,i-1,i-2}\right]\right) \\
=\ & K_{1n}^2 K_{2n} K_{3n} E\left(\sum_i\sum_j\sum_k\sum_l a_i a_j b_{k,k-1} c_{l,l-1,l-2}\right) \\
=\ & K_{1n}^2 K_{2n} K_{3n}\left[E\left(\sum_i\sum_k\sum_l a_i^2 b_{k,k-1} c_{l,l-1,l-2}\right) + 2E\left(\sum_{i<j}\sum_k\sum_l a_i a_j b_{k,k-1} c_{l,l-1,l-2}\right)\right] \\
\equiv\ & K_{1n}^2 K_{2n} K_{3n}\left[(\mathbf{1})+(\mathbf{2})\right].
\end{aligned}$$

By the independence and mean zero property of $a_i, a_j, b_{k,k-1}$ and $c_{l,l-1,l-2}$, the non zero contribution to $E\left(S_{n,1}^2 S_{n,2} U_n\right)$ are given by:

$$(\mathbf{1}) = \left(\sum_i E(a_i^2)\right)\left(\sum_i E[(b_{i+1,i} + b_{i,i-1} + b_{i-1,i-2} + b_{i-2,i-3})c_{i,i-1,i-2}]\right) + O_P(n^{-4}),$$

and

$$(\mathbf{2}) = 2 \times \left(\sum_i E(a_i b_{i,i-1} + a_{i-1} b_{i,i-1})\right)\left(\sum_i E[(a_i + a_{i-1} + a_{i-2})c_{i,i-1,i-2}]\right) + O_P(n^{-4}).$$

By tedious but simple algebra, we have

$$\begin{aligned}
&(\mathbf{1}) = (k_4 - k_2)\left(\sum_i (v_i^n)^2\right) \\
\times\ & \Bigg\{k_{\frac{4}{3}}\left(k_{\frac{7}{3}}^2 - k_1^2 k_{\frac{4}{3}}^2\right)\left(\sum_i (v_i^n)^{7/6}(v_{i-1}^n)^{7/6}(v_{i-2}^n)^{2/3} + \sum_i (v_i^n)^{2/3}(v_{i-1}^n)^{7/6}(v_{i-2}^n)^{7/6}\right) + \\
& k_1 k_{\frac{4}{3}}^2\left(k_{\frac{7}{3}} - k_1 k_{\frac{4}{3}}\right)\left(\begin{array}{c} \sum_i (v_i^n)^{2/3}(v_{i-1}^n)^{2/3}(v_{i-2}^n)^{7/6}(v_{i-3}^n)^{1/2} \\ + \sum_i (v_{i+1}^n)^{1/2}(v_i^n)^{7/6}(v_{i-1}^n)^{2/3}(v_{i-2}^n)^{2/3} \end{array}\right)\Bigg\} \\
& + O(n^{-4}),
\end{aligned}$$

and

$$\begin{aligned}
&(\mathbf{2}) = 2 \times \left\{k_1(k_3 - k_1 k_2)\left(\sum_i (v_i^n)^{3/2}(v_{i-1}^n)^{1/2} + \sum_i (v_i^n)^{1/2}(v_{i-1}^n)^{3/2}\right)\right\} \\
\times\ & \Bigg\{k_{\frac{4}{3}}^2 (k_{\frac{10}{3}} - k_2 k_{\frac{4}{3}}) \\
& \times \left(\sum_i (v_i^n)^{5/3}(v_{i-1}^n)^{2/3}(v_{i-2}^n)^{2/3} + \sum_i (v_i^n)^{2/3}(v_{i-1}^n)^{5/3}(v_{i-2}^n)^{2/3} + \sum_i (v_i^n)^{2/3}(v_{i-1}^n)^{2/3}(v_{i-2}^n)^{5/3}\right)\Bigg\} \\
& + O(n^{-4}).
\end{aligned}$$

Thus

$$E(S_{n,1}^2 S_{n,2} U_n) = K_{1n}^2 K_{2n} K_{3n}[(\mathbf{1}) + (\mathbf{2})] = \frac{\tau}{k_1^2 k_{\frac{4}{3}}^3} \frac{n^3}{V_n^{5/2}}[(\mathbf{1}) + (\mathbf{2})].$$

(a13) We have

$$\begin{aligned}
& E(S_{n,1} S_{n,2}^2 U_n) \\
= \ & K_{1n} K_{2n}^2 K_{3n} E\left(\left[\sum_i a_i\right]\left[\sum_i b_{i,i-1}\right]^2\left[\sum_i c_{i,i-1,i-2}\right]\right) \\
= \ & K_{1n} K_{2n}^2 K_{3n} E\left(\sum_i \sum_j \sum_k \sum_l b_{i,i-1} b_{j,j-1} a_k c_{l,l-1,l-2}\right) \\
= \ & K_{1n} K_{2n}^2 K_{3n}\left[E\left(\sum_i \sum_k \sum_l b_{i,i-1}^2 a_k c_{l,l-1,l-2}\right) + 2E\left(\sum_{i<j} \sum_k \sum_l b_{i,i-1} b_{j,j-1} a_k c_{l,l-1,l-2}\right)\right] \\
\equiv \ & K_{1n} K_{2n}^2 K_{3n}\left[(\mathbf{3}) + (\mathbf{4})\right].
\end{aligned}$$

By the independence and mean zero property of $a_i, a_j, b_{k,k-1}$ and $c_{l,l-1,l-2}$, the non zero contribution to $E(S_{n,1}S_{n,2}^2U_n)$ are given by:

$$(\mathbf{3}) \quad = \quad \left(\sum_i E(b_{i,i-1}^2)\right)\left(\sum_i E[(a_i + a_{i-1} + a_{i-2})c_{i,i-1,i-2}]\right) + O(n^{-4})$$

and

$$\begin{aligned}(\mathbf{4}) \quad = \quad & 2 \times \left(\sum_i E[b_{i,i-1}b_{i+1,i}]\right)\left(\sum_i E[(a_i + a_{i-1} + a_{i-2})c_{i,i-1,i-2}]\right) \\ & +2 \times \left(\sum_i E[b_{i,i-1}a_i + b_{i,i-1}a_{i-1}]\right) \times \\ & \quad \times \left(\sum_i E[b_{i,i-1}(c_{i-1,i-2,i-3} + c_{i,i-1,i-2} + c_{i+1,i,i-1} + c_{i+2,i+1,i})]\right) + O(n^{-4})\end{aligned}$$

By tedious but simple algebra, we have

$$\begin{aligned}(\mathbf{3}) \quad = \quad & \left(k_2^2 - k_1^4\right) k_{\frac{4}{3}}^2 \left(k_{\frac{10}{3}} - k_2 k_{\frac{4}{3}}\right) \sum_i v_i^n v_{i-1}^n \\ & \times \left(\sum_i (v_i^n)^{5/3}(v_{i-1}^n)^{2/3}(v_{i-2}^n)^{2/3} + \sum_i (v_i^n)^{2/3}(v_{i-1}^n)^{5/3}(v_{i-2}^n)^{2/3} + \sum_i (v_i^n)^{2/3}(v_{i-1}^n)^{2/3}(v_{i-2}^n)^{5/3}\right) \\ & +O(n^{-4})\end{aligned}$$

and

$$\begin{aligned}(\mathbf{4}) = \quad & 2 \times \left(k_1^2(k_2 - k_1^2)\sum_i (v_{i+1}^n)^{1/2} v_i^n (v_{i-1}^n)^{1/2}\right) \times \Big\{ k_{\frac{4}{3}}^2 (k_{\frac{10}{3}} - k_2 k_{\frac{4}{3}}) \\ & \times \left(\sum_i (v_i^n)^{5/3}(v_{i-1}^n)^{2/3}(v_{i-2}^n)^{2/3} + \sum_i (v_i^n)^{2/3}(v_{i-1}^n)^{5/3}(v_{i-2}^n)^{2/3} + \sum_i (v_i^n)^{2/3}(v_{i-1}^n)^{2/3}(v_{i-2}^n)^{5/3}\right)\Big\} \\ & +2 \times \left\{ k_1(k_3 - k_1k_2)\left(\sum_i (v_i^n)^{3/2}(v_{i-1}^n)^{1/2} + \sum_i (v_i^n)^{1/2}(v_{i-1}^n)^{3/2}\right)\right\} \\ & \times \Big\{ k_{\frac{4}{3}}(k_{\frac{7}{3}}^2 - k_1^2 k_{\frac{4}{3}}^2)\left(\sum_i (v_i^n)^{7/6}(v_{i-1}^n)^{7/6}(v_{i-2}^n)^{2/3} + \sum_i (v_{i+1}^n)^{2/3}(v_i^n)^{7/6}(v_{i-1}^n)^{7/6}\right) \\ & +k_1 k_{\frac{4}{3}}^2 (k_{\frac{7}{3}} - k_1 k_{\frac{4}{3}}) \left(\begin{array}{c} \sum_i (v_i^n)^{1/2}(v_{i-1}^n)^{7/6}(v_{i-2}^n)^{2/3}(v_{i-3}^n)^{2/3} \\ + \sum_i (v_{i+2}^n)^{2/3}(v_{i+1}^n)^{2/3}(v_i^n)^{7/6}(v_{i-1}^n)^{1/2} \end{array}\right)\Big\}.\end{aligned}$$

It follows that

$$E(S_{n,1}S_{n,2}^2U_n) = K_{1n}K_{2n}^2K_{3n}\left[(\mathbf{3}) + (\mathbf{4})\right] = \frac{\tau}{k_1^4 k_{\frac{4}{3}}^3}\frac{n^3}{V_n^{5/2}}[(\mathbf{3}) + (\mathbf{4})].$$

(a14) We have

$$
\begin{aligned}
& E(S_{n,2}^3 U_n) \\
= \; & K_{2n}^3 K_{3n} E\left(\left[\sum_i b_{i,i-1}\right]^3 \left[\sum_i c_{i,i-1,i-2}\right]\right) \\
= \; & K_{2n}^3 K_{3n} E\left(\sum_i \sum_j \sum_k \sum_l b_{i,i-1} b_{j,j-1} b_{k,k-1} c_{l,l-1,l-2}\right) \\
= \; & K_{2n}^3 K_{3n} \left[E\left(\sum_i \sum_l b_{i,i-1}^3 c_{l,l-1,l-2}\right) + 3E\left(\sum_{i<j} \sum_l b_{i,i-1}^2 b_{j,j-1} c_{l,l-1,l-2}\right) \right. \\
& \left. +3E\left(\sum_{i<j} \sum_l b_{i,i-1} b_{j,j-1}^2 c_{l,l-1,l-2}\right) + 6E\left(\sum_{i<j<k} \sum_l b_{i,i-1} b_{j,j-1} b_{k,k-1} c_{l,l-1,l-2}\right)\right] \\
= \; & K_{2n}^3 K_{3n} \left[(\mathbf{5}) + (\mathbf{6}) + (\mathbf{7}) + (\mathbf{8})\right].
\end{aligned}
$$

It is straightforward to see that

$$
\begin{aligned}
(\mathbf{5}) &= \sum_i E\left(b_{i,i-1}^3 [c_{i+2,i+1,i} + c_{i+1,i,i-1} + c_{i,i-1,i-2} + c_{i-1,i-2,i-3}]\right) = O(n^{-4}), \\
(\mathbf{8}) &= O(n^{-4}),
\end{aligned}
$$

and

$$
\begin{aligned}
& (\mathbf{6}) + (\mathbf{7}) \\
= \; & 3 \times \left(\sum_i E(b_{i,i-1}^2)\right) \left(\sum_i E[b_{i,i-1}(c_{i+2,i+1,i} + c_{i+1,i,i-1} + c_{i,i-1,i-2} + c_{i-1,i-2,i-3})]\right) + O(n^{-4}).
\end{aligned}
$$

The expansions lead to:

$$
\begin{aligned}
& (\mathbf{6}) + (\mathbf{7}) = 3 \times \left\{(k_2^2 - k_1^4) \sum_i v_i^n v_{i-1}^n\right\} \times \\
\times \; & \left\{ k_{\frac{4}{3}} \left(k_{\frac{7}{3}}^2 - k_1^2 k_{\frac{4}{3}}^2\right) \left(\sum_i (v_i^n)^{7/6} (v_{i-1}^n)^{7/6} (v_{i-2}^n)^{2/3} + \sum_i (v_{i+1}^n)^{2/3} (v_i^n)^{7/6} (v_{i-1}^n)^{7/6}\right) \right. \\
& \left. + k_1 k_{\frac{4}{3}}^2 \left(k_{\frac{7}{3}} - k_1 k_{\frac{4}{3}}\right) \left(\begin{array}{c} \sum_i (v_i^n)^{1/2} (v_{i-1}^n)^{7/6} (v_{i-2}^n)^{2/3} (v_{i-3}^n)^{2/3} \\ + \sum_i (v_{i+2}^n)^{2/3} (v_{i+1}^n)^{2/3} (v_i^n)^{7/6} (v_{i-1}^n)^{1/2} \end{array} \right) \right\} \\
& + O(n^{-4}).
\end{aligned}
$$

Hence

$$
E\left(S_{n,2}^3 U_n\right) = \frac{\tau}{k_1^6 k_{\frac{4}{3}}^3} \frac{n^3}{V_n^{5/2}} [(\mathbf{6}) + (\mathbf{7})].
$$

Proof of Lemma S2.5.

(a1) Follows directly given the definition of S_n and V_n.

(a2) Follows given parts (a2) and (a3) of Lemma S2.4.

(a3) Note that

$$E\left(S_n^3\right) = E\left(S_{n,1}^3\right) - 3E\left(S_{n,1}^2 S_{n,2}\right) + 3E\left(S_{n,1} S_{n,2}^2\right) - E\left(S_{n,2}^3\right).$$

The result follows by using parts (a7)-(a10) of Lemma S2.4 and (S2.8).

(a4) Write

$$E\left(S_n^3 U_n\right) = E\left(S_{n,1}^3 U_n\right) - 3E\left(S_{n,1}^2 S_{n,2} U_n\right) + 3E\left(S_{n,1} S_{n,2}^2 U_n\right) - E\left(S_{n,2}^3 U_n\right).$$

Then, the result follows by using parts (a11)-(a14) of Lemma S2.4 and (S2.8).

(a5) This follows given parts (a4)-(a6) of Lemma S2.4 and (S2.8).

Proof of Lemma S2.6. Proofs for (a1)-(a5) follow the same lines as in those of Lemma S2.5. The derivation are the same and we use Lemma 3.2 instead of (S2.8) to obtain the relevant probability limits. It remains to prove (a6). Since $V_n^* = O_P(1)$ with positive probability limit, we just have to show that conditionally on σ, $a_n \equiv \sqrt{n}\left(E^*(\hat{V}_n^*) - V_n^*\right) = O_P(1)$. For this it suffices to show that $E|a_n| = O(1)$ conditionally on σ. Using Lemma S1.2, we can see that for k_n large enough, we can see obtain:

$$\begin{aligned} a_n &= -(k_1^{-4}-1)n^{3/2}\sum_{j=1}^{n/k_n}(\hat{v}_j^n)^2 + \tau\left(n^{3/2}\sum_{j=1}^{n/k_n}(\hat{v}_j^n)^{4/3}(\hat{v}_{j-1}^n)^{2/3} + n^{3/2}\sum_{j=1}^{n/k_n}(\hat{v}_j^n)^{2/3}(\hat{v}_{j-1}^n)^{4/3}\right) \\ &\quad +2(2-k_1^{-2})\left(n^{3/2}\sum_{j=1}^{n/k_n}(\hat{v}_j^n)^{1/2}(\hat{v}_{j-1}^n)^{3/2} + n^{3/2}\sum_{j=1}^{n/k_n}(\hat{v}_j^n)^{3/2}(\hat{v}_{j-1}^n)^{1/2}\right) \\ &\quad -(k_1^{-4}-1)n^{3/2}\sum_{j=1}^{n/k_n}(\hat{v}_j^n)(\hat{v}_{j-1}^n), \end{aligned}$$

where $\hat{v}_j^n$'s involve returns in non overlapping blocks $j = 1, \ldots, n/k_n$. Hence, to conclude, it is sufficient to show that, conditionally on σ,

$$E\left(n^{3/2}\sum_{j=1}^{n/k_n}(\hat{v}_j^n)^2\right) = O(1) \quad \text{and} \quad E\left(n^{3/2}\sum_{j=1}^{n/k_n}(\hat{v}_j^n)^a(\hat{v}_{j-1}^n)^b\right) = O(1),$$

for $a, b > 0$ and $a+b = 2$. By definition, $\hat{v}_j^n = \frac{1}{k_n}\sum_{i=1}^{k_n} r_{i+(j-1)k_n}^2$ and thanks to the Jensen's inequality, we have: $E(\hat{v}_j^n)^a \leq \frac{1}{k_n}\sum_{i=1}^{k_n} E\left(|r_{i+(j-1)k_n}|^{2a}\right)$, for all $a \geq 1$. Using Eq. (2.1.34) of Jacod and Protter (2012), we can claim that, for all $p \geq 1$, $E\left(|r_i|^p\right) \leq \frac{K_p}{n^{p/2}}$. Thus, for some constant K_{2a},

$$E(\hat{v}_j^n)^a \leq \frac{K_{2a}}{n^a} \quad \text{for all} \quad a \geq 1. \tag{S2.10}$$

Also, if $0 < a < 1$, the Jensen's inequality implies that $E[(\hat{v}_j^n)^a] \leq [E(\hat{v}_j^n)]^a$ which, in turn and using (S2.10), is less or equal to $\frac{K_2^a}{n^a}$, for some constant K_2. This means that (S2.10) actually holds for all $a > 0$.

Since, conditionally on σ, r_i's are pairwise independent with $r_i \sim N(0, v_i^n)$, $\hat{v}_j^n$'s are also pairwise independent conditionally on σ. Hence, conditionally on σ,

$$E\left(n^{3/2}\sum_{j=1}^{n/k_n}(\hat{v}_j^n)^a(\hat{v}_{j-1}^n)^b\right) = n^{3/2}\sum_{j=1}^{n/k_n}E[(\hat{v}_j^n)^a]E[(\hat{v}_{j-1}^n)^b] \leq Cn^{3/2}\frac{n}{k_n}\frac{1}{n^a}\frac{1}{n^b} = C\frac{\sqrt{n}}{k_n} = O(1),$$

for some constant $C > 0$.

Appendix S3: Bootstrap test statistic for the log version of the jump test

The asymptotic test based on logarithm transformation of the linear version of the jump test as given by (6) has been proposed by Huang and Tauchen (2005). It follows from (4) and (5) that

$$\sqrt{n}\left(\log RV_n - \log BV_n\right) \xrightarrow{st} N\left(0, \tau \frac{IQ}{IV^2}\right), \quad \tau = \theta - 2,$$

and the test statistic of the log version of the jump test is given by

$$T_{\log,n} = \frac{\sqrt{n}\left(\log RV_n - \log BV_n\right)}{\sqrt{\tau \max\left(1, \frac{\widehat{IQ}_n}{BV_n^2}\right)}}.$$

To derive the bootstrap test statistic $T^*_{\log,n}$ for $T_{\log,n}$, we rely on the following result which is established as part of the proof of Theorem 3.1:

$$\Sigma_n^{*-1/2}\sqrt{n}\begin{pmatrix} RV_n^* - E^*\left(RV_n^*\right) \\ BV_n^* - E^*\left(BV_n^*\right) \end{pmatrix} \xrightarrow{d^*} N\left(0, I_2\right).$$

By a Taylor expansion, we have

$$\begin{aligned} \sqrt{n}\left(\log \tfrac{RV_n^*}{BV_n^*} - \log \tfrac{E^*(RV_n^*)}{E^*(BV_n^*)}\right) \quad &= \quad \begin{pmatrix} \frac{1}{E^*(RV_n^*)} & -\frac{1}{E^*(BV_n^*)} \end{pmatrix} \sqrt{n}\begin{pmatrix} RV_n^* - E^*(RV_n^*) \\ BV_n^* - E^*(BV_n^*) \end{pmatrix} \\ &\quad + o_{P^*}(1), \text{ Prob-}P. \end{aligned}$$

Conditionally on no jumps, $E^*(RV_n^*) \xrightarrow{P} IV$ and $E^*\left(BV_n^*\right) \xrightarrow{P} IV$. From (S1.1), we conclude that

$$\frac{\sqrt{n}\left(\log \frac{RV_n^*}{BV_n^*} - \log \frac{E^*(RV_n^*)}{E^*(BV_n^*)}\right)}{\sqrt{\tau \frac{IQ}{IV^2}}} \xrightarrow{d^*} N(0,1), \quad \text{in Prob-}P.$$

The bootstrap test statistic for $T_{\log,n}$ is given by

$$T^*_{\log,n} = \frac{\sqrt{n}\left(\log \frac{RV_n^*}{BV_n^*} - \log \frac{E^*(RV_n^*)}{E^*(BV_n^*)}\right)}{\sqrt{\tau \max\left(1, \frac{\widehat{IQ}_n^*}{(BV_n^*)^2}\right)}}.$$

$T_{\log,n}$ satisfies the conditions of Theorem 3.2 and if Condition A holds, this theorem applies and we can claim that $T^*_{\log,n}$ controls the strong asymptotic size and is alternative consistent.

Appendix S4: Bootstrap consistency for two alternative jump test statistics

The purpose of this section is to show that the local Gaussian bootstrap can be applied more generally than just to the BN-S test statistic. Specifically, we consider the jump test of Podolskij and Ziggel (hereafter PZ, 2010) and the jump test of Lee and Hannig (hereafter LH, 2010), which extends that of Lee and Mykland (2008). We first introduce the test statistics and their bootstrap versions. We then give a set of high level conditions on $\hat{v}_i^n$ under which the bootstrap is asymptotically valid when applied to each of these tests. The main result is Proposition S2.1, whose proof appears at the end of this section.

The PZ (2010) test statistic for jumps over $[0,1]$ is given by

$$T_n(p) = \frac{n^{\frac{p-1}{2}}\left(\sum_{i=1}^{n}|r_i|^p - \sum_{i=1}^{n}\xi_i|r_i|^p 1_{\{|r_i|\le cn^{-\varpi}\}}\right)}{\left(n^{p-1}Var(\xi_i)\sum_{i=1}^{n}|r_i|^{2p}1_{\{|r_i|\le cn^{-\varpi}\}}\right)^{\frac{1}{2}}}, \tag{S4.1}$$

where $p \ge 2$, ξ_i are positive i.i.d random variables, independent of X with $E(\xi_i) = 1$, $Var(\xi_i) > 0$ and, for some $a > 0$, $E(\xi_i^{2+a}) < \infty$; $c > 0$ and $\varpi \in (0, 1/2)$. PZ establish that for any $p \ge 2$, conditionally on Ω_0, i.e. when there are no jumps over $(0,1]$,

$$T_n(p) \xrightarrow{st} N(0,1) \tag{S4.2}$$

and, conditionally on Ω_1, i.e. under the occurrence of jumps over $(0,1]$,

$$T_n(p) \xrightarrow{P} \infty. \tag{S4.3}$$

LH's (2010) test statistic for jumps at a given date τ is given by

$$T_n(\tau) = \frac{\sqrt{n}r_i}{\hat{\sigma}(t_i)}, \tag{S4.4}$$

where i is such that $\tau \in (t_{i-1}, t_i]$ $(t_i = i/n)$ and

$$[\hat{\sigma}(t_i)]^2 = \frac{n}{\tilde{k}_n}\sum_{j=i-\tilde{k}_n}^{i-1} r_j^2 1_{\{|r_i|\le cn^{-\varpi}\}},$$

for some $c > 0$ and $\varpi \in (0, 1/2)$; and $\tilde{k}_n$ an arbitrary sequence of integers such that $\tilde{k}_n \to \infty$ and $\tilde{k}_n/n \to 0$ as $n \to \infty$. This test statistic is used to test whether there is a jump at a particular time $\tau \in (0,1]$. Let

$$\Omega_0^\tau = \{\omega : s \mapsto X_s(\omega) : \text{ is continuous at } s = \tau\}$$

and

$$\Omega_1^\tau = \{\omega : s \mapsto X_s(\omega) : \text{ is discontinuous at } s = \tau\}.$$

Conditionally on Ω_0^τ,

$$T_n(\tau) \xrightarrow{d} N(0,1), \tag{S4.5}$$

and conditionally on Ω_1^τ (see Theorem 1 of LH),

$$T_n(\tau) \xrightarrow{P} \infty. \tag{S4.6}$$

This test can also be used to detect the occurrence of big jumps over the whole interval $(0,1]$ by using critical value from the extreme value distribution μ, the asymptotic distribution of

$$T_n = \frac{\max_{1\le i\le n}|T_n(t_i)| - C_n}{S_n}, \tag{S4.7}$$

with $C_n = (2\log n)^{1/2} - \frac{\log\pi + \log(\log n)}{2(2\log n)^{1/2}}$, $S_n = \frac{1}{(2\log n)^{1/2}}$ and $\forall x \in \mathbb{R}$, $P(\mu \le x) = \exp(-e^{-x})$. We refer to LH (2010, p. 275) for the full description of the testing procedure for big jumps detection.

To define the bootstrap test statistics, let the local Gaussian bootstrap sample be $\{r_i^* : i = 1, \ldots, n\}$, with

$$r_i^* = \sqrt{\hat{v}_i^n} \cdot \eta_i,$$

where η_i is i.i.d $N(0,1)$ independent of the data and $\hat{v}_i^n$ is a local volatility estimator. We define the bootstrap version of PZ's test statistic $T_n(p)$ in (S4.1) by

$$T_n^*(p) = \frac{n^{\frac{p-1}{2}} \left(\sum_{i=1}^n |r_i^*|^p - \sum_{i=1}^n \zeta_i |r_i^*|^p 1_{\{|r_i^*| \leq cn^{-\varpi}\}} \right)}{\left(n^{p-1} Var(\zeta_i) \sum_{i=1}^n |r_i^*|^{2p} 1_{\{|r_i^*| \leq cn^{-\varpi}\}} \right)^{\frac{1}{2}}}, \tag{S4.8}$$

where ζ_i is an i.i.d sequence of positive random variables which is independent of the data, ξ_i and η_i, and has the same distribution as ξ_i.

The bootstrap version of LH's test statistic $T_n(\tau)$ in (S4.4) for jump at date $\tau \in (t_{i-1}, t_i]$ is

$$T_n^*(\tau) = \frac{\sqrt{n} r_i^*}{\hat{\sigma}^*(t_i)} = \frac{\sqrt{n \hat{v}_i^n}}{\hat{\sigma}^*(t_i)} \eta_i, \tag{S4.9}$$

where $[\hat{\sigma}^*(t_i)]^2$ is the bootstrap analog of $[\hat{\sigma}(t_i)]^2$, obtained by replacing r_m by r_m^* in $[\widehat{\sigma}(t_i)]^2$. More precisely,

$$[\hat{\sigma}^*(t_i)]^2 = \frac{n}{\tilde{k}_n} \sum_{j=i-\tilde{k}_n}^{i-1} r_j^{*^2} 1_{\{|r_j^*| \leq cn^{-\varpi}\}}.$$

We establish the bootstrap validity for the PZ and LH tests under the following Conditions A(PZ-p) and A(LH-τ), respectively. These conditions are the analogue to Condition A in the main text under which the validity of the local Gaussian bootstrap is established for the BN-S jump test.

Condition A(PZ-p)

(i) There exists $\delta > p$ such that,

$$\varpi < \frac{1}{2} - \frac{1}{2\delta} \quad \text{and} \quad \forall \ell \in (0, \delta], \quad n^{-1+\ell} \sum_{i=1}^n (\hat{v}_i^n)^\ell \xrightarrow{P} \int_0^1 \sigma_u^{2\ell} du.$$

(ii)

$$n^{-1+2p} \sum_{i=1}^n (\hat{v}_i^n)^{2p} = O_P(1).$$

Condition A(LH-τ)

(i) As $n \to \infty$, $\tilde{k}_n \to \infty$ and $\tilde{k}_n / n \to 0$.

(ii) For $j = i - \tilde{k}_n, \ldots, i-1$, $\quad n\hat{v}_j^n \xrightarrow{P} \sigma_\tau^2$ as $n \to \infty$ and $\frac{1}{\tilde{k}_n} \sum_{j=i-\widetilde{k}_n}^{i-1} \left(n\hat{v}_j^n \right)^2 = O_p(1)$, where i is such that $\tau \in (t_{i-1}, t_i]$.

Similarly to Condition A, these conditions apply to the sequence of local volatility estimates and so long as these conditions are satisfied both under the null and the alternative, the bootstrap test controls size under the null and is consistent under the alternative. The main result is as follows.

Proposition S4.1 *Let X be an Itô semimartingale defined by (1) and satisfying Assumption (H-2).*

(i) *If Condition A(PZ-p) holds for some $p \geq 2$, then*

$$T_n^*(p) \xrightarrow{d^*} N(0,1),$$

in probability.

(ii) *If Condition A(LH-τ) holds for some $\tau \in (0,1]$, then*

$$T_n^*(\tau) \xrightarrow{d^*} N(0,1),$$

in probability.

Before providing the proof of this proposition, we make the following remarks.

Remark 1 *The bootstrap version $T_n^*(\tau)$ of LH's test statistic can be used to detect the occurrence of big jumps over a fixed time interval such as $[0,1]$. A sufficient condition is that Condition A(LH-τ) holds for all $\tau \in (0,1]$. Under this condition and because the η_i's are independent conditionally on the data, the $T_n^*(t_i)$ are also conditionally independent and asymptotically standard normal over the entire interval. In this case, the bootstrap can be used to compute the critical values of LH's (2010) big jump test. In particular, we reject the absence of jumps at any given time t_i whenever the absolute value of $T_n(t_i)$ exceeds $q_\alpha^* S_n + C_n$, where q_α^* is the α quantile of the bootstrap distribution of*

$$T_n^* = \frac{\max_{1 \leq i \leq n} |T_n^*(t_i)| - C_n}{S_n}. \tag{S4.10}$$

Note that the test statistic T_n can be used directly to test for occurrence of jumps over long time intervals. This has been highlighted by Aït-Sahalia and Jacod (2014, Chap. 10). In this case, critical values from the extreme value distribution μ or from the bootstrap distribution of T_n^* can be used.

Remark 2 *A bootstrap version of the QQ-plot test for small jumps can also be obtained by comparing the empirical quantiles of $\{T_n(t_i) : i = \tilde{k}_n, \ldots, n\}$ to those of the bootstrap samples $\{T_n^*(t_i) : i = \tilde{k}_n, \ldots, n\}$. The bootstrap samples replace the artificial samples drawn from the standard normal distribution originally proposed by Lee and Hannig (2010).*

Remark 3 *If we implement the bootstrap jump tests of PZ and LH with $\hat{v}_i^n$ based on thresholding (as discussed in Section 3.2 for the BN-S test), the truncation parameter, say ϖ', used for the bootstrap data generating process need not be equal to ϖ - the truncation parameter used in the test statistics. In particular, to satisfy Condition A(PZ-p), one can first choose $\varpi \in (0,1/2)$ and then, set ϖ' such that $\max\left(\frac{2p-1}{4p-r}, \frac{\delta-1}{2\delta-r}\right) \leq \varpi' < \frac{1}{2}$ for some $\delta > \max\left(p, \frac{1}{1-2\varpi}\right)$. We can show that under these conditions, Condition A(PZ-p) holds by Theorem 9.4.1 of Jacod and Protter (2012). While these restrictions on ϖ and ϖ' matter for Condition A(PZ-p) to be satisfied under the alternative of occurrence of jumps, they are immaterial under the null of no jumps since this condition is fulfilled for any choice ϖ and ϖ' in $(0,1/2)$. Given Theorem 9.3.2 of Jacod and Protter (2012) and their comments leading to that theorem, we can claim that Condition A(LH-τ) is also fulfilled for a local volatility estimate based on thresholding so long as we maintain that the volatility process σ_s^2 is continuous at $s = \tau$. The validity of the test over the full range $[0,1]$ is therefore guaranteed under the common assumption that the price and volatility processes do not jump at the same time.*

Remark 4 *We have implemented the bootstrap versions of the PZ (2010) and LH (2010) tests in unreported simulation results using the same data generating processes as in the main text. Our findings are as follows: (1) the PZ (2010) is slightly oversized under the null of no jumps and the local Gaussian* *bootstrap helps alleviate these finite size distortions without sacrificing power; (2) the bootstrap big-jump LH (2010) test outperforms the original test of LH (2010) by showing a lower probability of global misclassification. Specifically, the bootstrap version of the test has a lower probability of global spurious detection of jumps than the original test while both tests have the same probability of global failure to detect jumps. (3) the small-jump LH test also has a low probability of global spurious detection of jumps that ranges between between 3.65% for* $n = 78$ *and 3.47% for* $n = 576$*, although not as low as that of the bootstrap big-jump LH (2010). Its probability of global success in detecting actual jumps is much smaller than both versions of the big-jump LH test when the alternative is finite activity jumps but it dominates any of these two tests when there are infinite activity jumps. This is as expected since the small-jump test of LH (2010) is especially designed to detect small jumps; (4) The big-jump test over long time intervals using directly* (S4.7) *(Aït-Sahalia and Jacod (2014, Chap. 10)) has a large size distortion under the null of no jumps that decays slowly from 63.14% (*$n = 78$*) to 58.09% (*$n = 576$*). Interestingly, the bootstrap version of this test, with rejection rates under the null from 3.14% to 5.56%, corrects this size distortion while showing reasonable power.*

Proof of Proposition S4.1: (i) Let

$$a_n^* = n^{\frac{p-1}{2}} \left(\sum_{i=1}^n |r_i^*|^p - \sum_{i=1}^n \zeta_i |r_i^*|^p 1_{\{|r_i^*| \le c n^{-\varpi}\}} \right), \quad d_n^* = n^{p-1} Var(\zeta_i) \sum_{i=1}^n |r_i^*|^{2p} 1_{\{|r_i^*| \le c n^{-\varpi}\}},$$

so that $T_n^*(p) = a_n^* / \sqrt{d_n^*}$. Let

$$a_{1n}^* = n^{\frac{p-1}{2}} \sum_{i=1}^n |r_i^*|^p (1 - \zeta_i), \quad a_{2n}^* = n^{\frac{p-1}{2}} \sum_{i=1}^n \zeta_i |r_i^*|^p 1_{\{|r_i^*| \le c n^{-\varpi}\}}.$$

Clearly, $a_n^* = a_{1n}^* + a_{2n}^*$. We will show that (a) $a_{2n}^* = o_{P^*}(1)$, in probability, and that d_n^* is positive with probability approaching one so that $T_n^*(p) = a_{1n}^* / \sqrt{d_n^*} + o_{P^*}(1)$. Let $v_n = Var^*(a_{1n}^*)$ and $S_n^* = a_{1n}^* / \sqrt{v_n}$. We complete the proof of statement (i) by showing that: (b) $S_n^* \xrightarrow{d^*} N(0,1)$, in probability and (c) $d_n^* - v_n \xrightarrow{P^*} 0$, in probability.

(a) We have

$$|a_{2n}^*| \le n^{\frac{p-1}{2}} \sum_{i=1}^n |\zeta_i| |r_i^*|^p 1_{\{|r_i^*| \le c n^{-\varpi}\}} \le n^{\frac{p-1}{2}} \sum_{i=1}^n \zeta_i |r_i^*|^p \frac{|r_i^*|^l}{c^l n^{-\varpi l}},$$

for all $l > 0$. Hence, using the fact that $r_i^* = \sqrt{\hat{v}_i^n} \cdot \eta_i$, $\eta_i \sim N(0,1)$, we have that

$$E^* |a_{2n}^*| \le C \cdot n^{\frac{p-1}{2} + l\varpi} \sum_{i=1}^n (\hat{v}_i^n)^{\frac{p+l}{2}} = C \cdot n^{\frac{1}{2} + l\varpi - \frac{l}{2}} \cdot n^{-1 + \frac{p+l}{2}} \sum_{i=1}^n (\hat{v}_i^n)^{\frac{p+l}{2}},$$

where $C > 0$ is a generic constant. Choosing l close enough to δ so that $\varpi < \frac{1}{2} - \frac{1}{2l}$ ensures, using Condition A(PZ-p)-(i) that $a_{2n}^* = o_{P^*}(1)$ in probability. The positivity of d_n^* is proven in part (c) below.

(b) Simple calculations show that $v_n = \mu_{2p} Var(\zeta_i) n^{-1+p} \sum_{i=1}^n (\hat{v}_i^n)^p$ which, under Condition A(PZ-p)-(i) converges in probability to the almost surely positive random variable $\mu_{2p} Var(\zeta_i) \int_0^1 \sigma_u^{2p} du$. We can therefore focus on establishing the conditions that ensure that a_{1n}^* is asymptotically normal. Note that

$$a_{1n}^* = \sum_{i=1}^n n^{\frac{p-1}{2}} |r_i^*|^p (1 - \zeta_i) = \sum_{i=1}^n n^{\frac{p-1}{2}} (\hat{v}_i^n)^{\frac{p}{2}} |\eta_i|^p (1 - \zeta_i).$$

By the independence across i of the terms in this summation and the fact that $E^*(|\eta_i|^p(1-\zeta_i)) = 0$, it suffices to verify the Lyapunov condition to conclude (b). That is, we show that there exists $\nu > 0$ such that

$$\sum_{i=1}^{n} E^* \left| n^{\frac{p-1}{2}} |r_i^*|^p (1-\zeta_i) \right|^{2+\nu} = o_{P^*}(1),$$

in probability. It is not hard to see that

$$\sum_{i=1}^{n} E^* \left| n^{\frac{p-1}{2}} |r_i^*|^p (1-\zeta_i) \right|^{2+\nu} = C \cdot n^{-\nu/2} \cdot n^{-1+p(1+\nu/2)} \sum_{i=1}^{n} (\hat{v}_i^n)^{p(1+\nu/2)}.$$

Again, one can choose $\nu > 0$ so that $p < p(1+\nu/2) \leq \delta$ and use Condition A(PZ-p)-(i) to conclude.

(c) Note that

$$d_n^* = n^{p-1} Var(\zeta_i) \sum_{i=1}^{n} |r_i^*|^{2p} - n^{p-1} Var(\zeta_i) \sum_{i=1}^{n} |r_i^*|^{2p} 1_{(|r_i^*| > cn^{-\varpi})} \equiv d_{1n}^* + d_{2n}^*.$$

It is not hard to prove by following similar steps as those in (a) above that $d_{2n}^* = o_{P^*}(1)$ in probability. Hence, it suffices to show that $d_{1n}^* - v_n \xrightarrow{P^*} 0$, in probability. Note that $E^*(d_{1n}^*) = v_n$ and it suffices to show that $Var^*(d_{1n}^*) = o_P(1)$ to conclude (d). We have:

$$Var^*(d_{1n}^*) = Var\left(|\eta_i|^{2p}\right) [Var(\zeta_i)]^2 n^{-1} n^{-1+2p} \sum_{i=1}^{n} (\hat{v}_i^n)^{2p} = o_P(1),$$

thanks to Condition A(PZ-p)-(ii). The positivity of d_n^* also follows.

(ii) Since $\eta_i \sim N(0,1)$ and is independent of the data, it suffices to show that $\frac{\sqrt{n\hat{v}_i^n}}{\hat{\sigma}^*(t_i)} \xrightarrow{P^*} 1$ in probability and, under Condition A(LH-τ), it suffices to show that $[\hat{\sigma}^*(t_i)]^2 \xrightarrow{P^*} \sigma_\tau^2$, in probability. For this, we show that

$$E^*\left([\hat{\sigma}^*(t_i)]^2 - \sigma_\tau^2\right) \xrightarrow{P} 0 \text{ and } Var^*\left([\hat{\sigma}^*(t_i)]^2 - \sigma_\tau^2\right) \xrightarrow{P} 0.$$

The following inequality (proven by successive applications of the Hölder inequality with Hölder conjugates $q/p > 1$ and $q/(q-p)$) and the Markov inequality (with exponent $2q$) implies that for any $q > p > 0$,

$$E^*\left(|\sqrt{n} r_j^*|^{2p} 1_{\{|r_j^*| \leq cn^{-\varpi}\}}\right) \leq K \left(n\hat{v}_j^n\right)^q n^{-2(q-p)(1/2-\varpi)}, \tag{S4.11}$$

where K is a positive constant. To show that $E^*\left([\hat{\sigma}^*(t_i)]^2\right) \xrightarrow{P} \sigma_\tau^2$, it suffices to show that

$$E^*\left(\frac{n}{\tilde{k}_n} \sum_{j=i-\tilde{k}_n}^{i-1} r_j^{*^2} 1_{\{|r_j^*| \leq cn^{-\varpi}\}}\right) \xrightarrow{P} 0,$$

since

$$E^*\left(\frac{n}{\tilde{k}_n} \sum_{j=i-\tilde{k}_n}^{i-1} r_j^{*^2}\right) = \frac{1}{\tilde{k}_n} \sum_{j=i-\tilde{k}_n}^{i-1} (n\hat{v}_j^n) \xrightarrow{P} \sigma_\tau^2.$$

Note that

$$
\begin{aligned}
E^*\left(\frac{n}{\tilde{k}_n}\sum_{j=i-\tilde{k}_n}^{i-1} r_j^{*2} 1_{\{|r_j^*|\le cn^{-\varpi}\}}\right) &= \frac{1}{\tilde{k}_n}\sum_{j=i-\tilde{k}_n}^{i-1} E^*\left((\sqrt{n}r_j^*)^2 1_{\{|r_j^*|\le cn^{-\varpi}\}}\right) \\
&\le K \underbrace{\left(\frac{1}{\tilde{k}_n}\sum_{j=i-\tilde{k}_n}^{i-1}\left(n\hat{v}_j^n\right)^q\right)}_{=O_p(1)} \underbrace{n^{-2(q-1)(1/2-\varpi)}}_{=o(1)},
\end{aligned}
$$

where the above inequality follows given (S4.11) with $q > p = 1$. Next, we show that $Var^*\left(\left[\hat{\sigma}^*\left(t_i\right)\right]^2\right) \xrightarrow{P} 0$. It is not hard to obtain that

$$
\begin{aligned}
Var^*\left(\left[\hat{\sigma}^*\left(t_i\right)\right]^2\right) &= \frac{1}{\tilde{k}_n^2}\sum_{j=i-\tilde{k}_n}^{i-1}\left(n\hat{v}_j^n\right)^2 \cdot Var^*\left(\eta_j^2 1_{\{|r_j^*|\le cn^{-\varpi}\}}\right) \\
&\le \frac{1}{\tilde{k}_n^2}\sum_{j=i-\tilde{k}_n}^{i-1}\left(n\hat{v}_j^n\right)^2 \cdot E^*\left(\eta_j^4\right) \le K \underbrace{\frac{1}{\tilde{k}_n}}_{=o(1)} \underbrace{\left(\frac{1}{\tilde{k}_n}\sum_{j=i-\tilde{k}_n}^{i-1}\left(n\hat{v}_j^n\right)^2\right)}_{=O_p(1)}.
\end{aligned}
$$

The desired result follows from Condition A(LH-τ).

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