Appendix for "Disappointment Aversion, Long-Run Risks and Aggregate Asset Prices"

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A Reproducing the Bansal and Yaron (2004) Model with a Markov-Switching Model

We start with the LRR model of BY for the endowment[1]:

$$\begin{aligned}
\Delta c_{t+1} &= x_t + \sigma_t \epsilon_{c,t+1} \\
\Delta d_{t+1} &= (1-\phi_d)\,\mu_x + \phi_d x_t + \nu_d \sigma_t \epsilon_{d,t+1} \\
x_{t+1} &= (1-\phi_x)\,\mu_x + \phi_x x_t + \nu_x \sigma_t \epsilon_{x,t+1} \\
\sigma^2_{t+1} &= (1-\phi_\sigma)\,\mu_\sigma + \phi_\sigma \sigma^2_t + \nu_\sigma \epsilon_{\sigma,t+1}
\end{aligned} \tag{A.1}$$

where

$$\begin{pmatrix} \epsilon_{c,t+1} \\ \epsilon_{d,t+1} \\ \epsilon_{x,t+1} \\ \epsilon_{\sigma,t+1} \end{pmatrix} \mid J_t \sim \mathcal{NID}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho_1 & 0 & 0 \\ \rho_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right).$$

The main features of this multivariate process are:

1. The expected means of the consumption and dividend growth rates are a linear function of the same autoregressive process of order one denoted x_t;
2. The conditional variances of the consumption and dividend growth rates are a linear function of the same autoregressive process of order one denoted σ_t^2;
3. The variables x_{t+1} and σ^2_{t+1} are independent conditionally to past information;
4. The innovations of the consumption and dividend growth rates are correlated conditionally to past information.

Our goal here is to characterize a Markov Switching (MS) model:

$$\begin{aligned}
\Delta c_{t+1} &= \mu_c(s_t) + (\omega_c(s_t))^{1/2}\,\varepsilon_{c,t+1} \\
\Delta d_{t+1} &= \mu_d(s_t) + (\omega_d(s_t))^{1/2}\,\varepsilon_{d,t+1},
\end{aligned} \tag{A.2}$$

where $\varepsilon_{c,t+1}$ and $\varepsilon_{d,t+1}$ follow a bivariate standard normal process with mean zero and correlation ρ_1. The MS model (A.2) will have the same features as the original endoment model (A.1).

For the MS model, the first characteristic of the Bansal and Yaron (2004) model implies that one has to assume that the expected means of the consumption and dividend growth rates are a linear function of the same Markov chain with two states given that a two-state Markov chain is an AR(1) process. Likewise, the second one implies that the conditional variances of the consumption and dividend growth rates are a linear function of the same

[1]Note that in the original modelof BY the parameter $\rho_1$was zero.

two-state Markov chain. According to the third characteristic, the two Markov chains should be independent. Consequently, we should assume that the Markov chain s_t has 4 states, two states for the conditional mean and two states for the conditional variance, and that the transition matrix P is restricted such as the conditional mean and the conditional variance are independent. Finally, the last characteristic implies that the correlation vector in the four states is equal to $(\rho_1, \rho_1, \rho_1, \rho_1)^\top$.

We would like to match an AR(1) process, say z_t, like x_t or σ_t^2 by a two-state Markov chain, say z_t^*. We assume that $z_t^* = a + by_t$ where y_t is a two-state Markov chain that takes the values 0 (first state) and 1 (second state), and where the transition matrix P_y of y_t is given by:

$$P_y^\top = \begin{pmatrix} p_{y,11} & 1 - p_{y,11} \\ 1 - p_{y,22} & p_{y,22} \end{pmatrix}.$$

The (unconditional) stationary distribution of y_t is

$$\pi_{y,1} = Prob\,(y_t = 0) = \frac{1 - p_{y,22}}{2 - p_{y,11} - p_{y,22}}, \quad \pi_{y,2} = Prob\,(y = 1) = \frac{1 - p_{y,11}}{2 - p_{y,11} - p_{y,22}}. \tag{A.3}$$

Without loss of generality, we assume that $b > 0$, i.e. the second state corresponds to the highest value of z_t^*.

Our goal is to find the vector $\theta = (p_{y,11}, p_{y,22}, a, b)^\top$ so that moments of the two-state Markov chain z_t^* are equal to moments of the AR(1) process z_t. The first moments we want to match are the mean, the variance and the first-order autocorrelation of the AR(1) process z_t denoted μ_z, σ_z^2 and ϕ_z respectively. Given that the dimension of the vector θ is four, another restriction is needed. For instance, Mehra and Prescott (1985) assumed $p_{y,11} = p_{y,22}$. In contrast, we will focus on matching the kurtosis of the process z_t denoted k_z. We will show below that matching the mean, the variance, the kurtosis and the first-order autocorrelation does not fully identify the parameters. However, knowing the sign of the skewness of z_t (denotes s_z) and the four other moments will fully identify the vector θ.

The moments of the AR(1) process z_t are related to those of the two-state Markov chain y_t as follows:

$$\begin{aligned}
\mu_z &= a + b\mu_y = a + b\pi_{y,2} \\
\sigma_z^2 &= b^2\sigma_y^2 = b^2\pi_{y,1}\pi_{y,2} \\
\phi_z &= \rho_y = p_{y,11} + p_{y,22} - 1 \\
k_z &= k_y = \frac{\pi_{y,1}^2}{\pi_{y,2}} + \frac{\pi_{y,2}^2}{\pi_{y,1}} \\
s_z &= s_y = \frac{\pi_{y,1} - \pi_{y,2}}{\sqrt{\pi_{y,1}\pi_{y,2}}}
\end{aligned} \tag{A.4}$$

Equation (A.4) combined with (A.3) characterizes the moments of the AR(1) process in terms of the vector θ. As pointed out above, Mehra and Prescott (1985) assumed that $p_{y,11} = p_{y,22}$, which implies $s_z = 0$ and $k_z = 1$. The empirical evidence reported in Cecchetti, Lam and Mark (1990) suggests that the kurtosis of consumption growth is higher than one and that its skewness is negative.[2] We will now invert the characterization (A.4), that is, we will determine the vector θ in terms of the moments of z_t.

The vector θ of parameters of the two-state Markov chain that matches the AR(1) process z_t is given by:

$$\begin{aligned}
p_{y,11} &= \frac{1+\phi_z}{2} - \frac{1-\phi_z}{2}\sqrt{\frac{k_z-1}{k_z+3}},\ p_{y,22} = \frac{1+\phi_z}{2} + \frac{1-\phi_z}{2}\sqrt{\frac{k_z-1}{k_z+3}} \text{ if } s_z \leq 0, \\
p_{y,11} &= \frac{1+\phi_z}{2} + \frac{1-\phi_z}{2}\sqrt{\frac{k_z-1}{k_z+3}},\ p_{y,22} = \frac{1+\phi_z}{2} - \frac{1-\phi_z}{2}\sqrt{\frac{k_z-1}{k_z+3}} \text{ if } s_z > 0, \\
b &= \frac{\sigma_z}{\sqrt{\pi_{y,1}\pi_{y,2}}},\ a = \mu_z - b\pi_{y,2}
\end{aligned} \tag{A.5}$$

and $\pi_{y,1}$ and $\pi_{y,2}$ are connected to $p_{y,11}$ and $p_{y,22}$ through (A.3).

The mean μ_x and the first autocorrelation ϕ_x of x_t, and the mean μ_σ and the first autocorrelation ϕ_σ of σ_t^2 are given in (A.1). The variance, the skewness and the kurtosis of x_t and σ_t^2 are given by:

$$\begin{aligned}
\sigma_x^2 &= \frac{\nu_x^2\mu_\sigma}{1-\phi_x^2},\ s_x = 0,\ k_x = 3\frac{\left(1-\phi_x^2\right)^2}{1-\phi_x^4}\left(1 + 2\frac{\phi_x^2}{1-\phi_x^2}\frac{\phi_\sigma}{\mu_\sigma} + \frac{\nu_\sigma^2}{\mu_\sigma\left(1-\phi_\sigma^2\right)}\right) \\
\sigma_\sigma^2 &= \frac{\nu_\sigma^2}{1-\phi_\sigma^2},\ s_\sigma = 0,\ k_\sigma = 3.
\end{aligned} \tag{A.6}$$

Observe that the skewness of the conditional mean of consumption growth equals zero in Bansal and Yaron (2004) as in Mehra and Prescott (1985). In contrast, in order to generate a kurtosis higher than one, the Markov switching needs some skewness. Given that the skewness of consumption growth is empirically negative, we make this identification assumption, that is, we use the first line in (A.5) to identify the transition probabilities $p_{x,11}$ and $p_{x,22}$.

Likewise, the skewness of the conditional variance is zero in Bansal and Yaron (2004), somewhat unrealistic given that the variance is a positive random variable. A popular variance model is the Heston (1993) model where the stationary distribution of the variance process is a Gamma distribution. Given that the skewness of a Gamma distribution is positive, we make the same assumption on σ_t^2 and therefore, the second line in (A.5) to identify the transition probabilities $p_{\sigma,11}$ and $p_{\sigma,22}$.

[2]Strictly speaking, the process x_t here is the expected mean of the consumption growth and not the growth. Therefore, the skewness and kurtosis of these two processes are different but connected.

We have now the two independent Markov chains that generate the expected mean and variance of consumption growth. Putting together these two processes leads to a four-state Markov chain (low mean and low variance, low mean and high variance, high mean and low variance, high mean and high variance) whose transition probability matrix is given by:

$$P^{\top}=\left[\begin{array}{cccc} p_{x,11}p_{\sigma,11} & p_{x,11}p_{\sigma,12} & p_{x,12}p_{\sigma,11} & p_{x,12}p_{\sigma,12} \\ p_{x,11}p_{\sigma,21} & p_{x,11}p_{\sigma,22} & p_{x,12}p_{\sigma,21} & p_{x,12}p_{\sigma,22} \\ p_{x,21}p_{\sigma,11} & p_{x,21}p_{\sigma,12} & p_{x,22}p_{\sigma,11} & p_{x,11}p_{\sigma,12} \\ p_{x,21}p_{\sigma,21} & p_{x,21}p_{\sigma,22} & p_{x,22}p_{\sigma,21} & p_{x,22}p_{\sigma,22} \end{array}\right] \tag{A.7}$$

where $p_{\cdot,12}=1-p_{\cdot,11}$ and $p_{\cdot,21}=1-p_{\cdot,22}$. The vectors μ_c, ω_c, μ_d, and ω_d defined in (A.2) are given by:

$$\begin{array}{l} \mu_c=\left(a_x,a_x,a_x+b_x,a_x+b_x\right)^{\top} \\ \omega_c=\left(a_\sigma,a_\sigma+b_\sigma,a_\sigma,a_\sigma+b_\sigma\right)^{\top} \\ \mu_d=\left(1-\phi_d\right)\mu_x e+\phi_d\mu_c \\ \omega_d=\nu_d^2\omega_c. \end{array} \tag{A.8}$$

where $e=(1,1,1,1)^{\top}$.

B Proofs of Formulas for Asset Prices

The formulas are proved using particular properties of Markov switching processes. It is well kown that (see, e.g., Hamilton (1994), page 679):

$$\forall h,\ E\left[\zeta_{t+h}\mid J_t\right]=P^h\zeta_t,\quad\text{and}\quad P^h\Pi=\Pi. \tag{A.9}$$

Also, for any vectors $a,b\in\mathbb{R}^N$, we have:

$$\left(a^{\top}\zeta_t\right)\left(b^{\top}\zeta_t\right)=\left(a\odot b\right)^{\top}\zeta_t, \tag{A.10}$$

In addition, we will need the following Lemma.
Lemma 0: *Given two standard normal random variables ϵ_1 and ϵ_2 with correlation ρ, and three real numbers x_1, σ_1 and σ_2, one has:*

$$E\left[\exp\left(\sigma_1\epsilon_1+\sigma_2\epsilon_2\right)\mathbf{1}\left(\epsilon_1<x_1\right)\right]=\exp\left(\frac{1}{2}\left(\sigma_1^2+2\rho\sigma_1\sigma_2+\sigma_2^2\right)\right)\Phi\left(x_1-\left(\sigma_1+\rho\sigma_2\right)\right) \tag{A.11}$$

where Φ denotes the cumulative distribution function of the standard normal, and $\mathbf{1}(\cdot)$ denotes the indicator function.

B.1 Utility-Consumption Ratios

B.1.1 First utility-consumption ratio

Recall that the GDA certainty equivalent may be written:

$$\mathcal{R}_t\left(V_{t+1}\right)=\left(E\left[\frac{I_{\alpha,1}\left(\frac{V_{t+1}}{\kappa\mathcal{R}_t(V_{t+1})}\right)}{E\left[I_{\alpha,\kappa}\left(\frac{V_{t+1}}{\kappa\mathcal{R}_t(V_{t+1})}\right)\mid J_t\right]}V_{t+1}^{1-\gamma}\mid J_t\right]\right)^{\frac{1}{1-\gamma}} \tag{A.12}$$

where

$$I_{\alpha,y}\left(x\right)=1+\left(\frac{1}{\alpha}-1\right)y^{1-\gamma}\mathbf{1}(x<1).$$

Dividing each side by C_t, it follows from (A.12) that

$$\frac{\mathcal{R}_t\left(V_{t+1}\right)}{C_t}=\left(E\left[\frac{I_{\alpha,1}\left(\frac{V_{t+1}}{\kappa\mathcal{R}_t(V_{t+1})}\right)}{E\left[I_{\alpha,\kappa}\left(\frac{V_{t+1}}{\kappa\mathcal{R}_t(V_{t+1})}\right)\mid J_t\right]}\left(\frac{V_{t+1}}{C_{t+1}}\right)^{1-\gamma}\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma}\mid J_t\right]\right)^{\frac{1}{1-\gamma}} \tag{A.13}$$

or

$$\lambda_{1z}^{\top}\zeta_t=\left(E\left[\frac{I_{\alpha,1}\left(\frac{V_{t+1}}{\kappa\mathcal{R}_t(V_{t+1})}\right)\left(\lambda_{1v}^{\top}\zeta_{t+1}\right)^{1-\gamma}\exp\left(\left(1-\gamma\right)\Delta c_{t+1}\right)}{E\left[I_{\alpha,\kappa}\left(\frac{V_{t+1}}{\kappa\mathcal{R}_t(V_{t+1})}\right)\mid J_t\right]}\mid J_t\right]\right)^{\frac{1}{1-\gamma}} \tag{A.14}$$

Notice that one has:

$$\frac{V_{t+1}}{\kappa\mathcal{R}_t\left(V_{t+1}\right)}=\frac{1}{\kappa}\frac{V_{t+1}}{C_{t+1}}\frac{C_t}{\mathcal{R}_t\left(V_{t+1}\right)}\frac{C_{t+1}}{C_t}=\frac{1}{\kappa}\frac{\lambda_{1v}^{\top}\zeta_{t+1}}{\lambda_{1z}^{\top}\zeta_t}\exp\left(\Delta c_{t+1}\right)$$

and that

$$\frac{V_{t+1}}{\kappa\mathcal{R}_t\left(V_{t+1}\right)}<1\qquad\Leftrightarrow\qquad\varepsilon_{c,t+1}<\frac{\ln\left(\kappa\frac{\lambda_{1z}^{\top}\zeta_t}{\lambda_{1v}^{\top}\zeta_{t+1}}\right)-\mu_c^{\top}\zeta_t}{\left(\omega_c^{\top}\zeta_t\right)^{1/2}}.$$

Then, the denominator in (A.14) is given by:

$$E\left[I_{\alpha,\kappa}\left(\frac{V_{t+1}}{\kappa\mathcal{R}_t\left(V_{t+1}\right)}\right)\mid J_t\right]=1+\left(\alpha^{-1}-1\right)\kappa^{1-\gamma}E\left[\mathbf{1}\left(\varepsilon_{c,t+1}<\frac{\ln\left(\kappa\frac{\lambda_{1z}^{\top}\zeta_t}{\lambda_{1v}^{\top}\zeta_{t+1}}\right)-\mu_c^{\top}\zeta_t}{\left(\omega_c^{\top}\zeta_t\right)^{1/2}}\right)\mid J_t\right]$$

where we have

$$
\begin{aligned}
& E\left[\mathbf{1}\left(\varepsilon_{c,t+1}<\frac{\ln\left(\kappa\frac{\lambda_{1z}^{\top}\zeta_t}{\lambda_{1v}^{\top}\zeta_{t+1}}\right)-\mu_c^{\top}\zeta_t}{\left(\omega_c^{\top}\zeta_t\right)^{1/2}}\right)\mid J_t\right] \\
&\quad = E\left[E\left[\mathbf{1}\left(\varepsilon_{c,t+1}<\frac{\ln\left(\kappa\frac{\lambda_{1z}^{\top}\zeta_t}{\lambda_{1v}^{\top}\zeta_{t+1}}\right)-\mu_c^{\top}\zeta_t}{\left(\omega_c^{\top}\zeta_t\right)^{1/2}}\right)\mid\{\zeta_\tau,\tau\in\mathbb{Z}\},J_t\right]\mid J_t\right] \\
&\quad = E\left[\Phi\left(\frac{\ln\left(\kappa\frac{\lambda_{1z}^{\top}\zeta_t}{\lambda_{1v}^{\top}\zeta_{t+1}}\right)-\mu_c^{\top}\zeta_t}{\left(\omega_c^{\top}\zeta_t\right)^{1/2}}\right)\mid J_t\right]=E\left[\Phi\left(\frac{\ln\left(\kappa\frac{\lambda_{1z}^{\top}\zeta_t}{\lambda_{1v}^{\top}\zeta_{t+1}}\right)-\mu_c^{\top}\zeta_t}{\left(\omega_c^{\top}\zeta_t\right)^{1/2}}\right)\mid s_t\right] \\
&\quad = \sum_{j=1}^{N}p_{s_t,j}\Phi\left(\frac{\ln\left(\kappa\frac{\lambda_{1z,s_t}}{\lambda_{1v,j}}\right)-\mu_{c,s_t}}{\omega_{c,s_t}^{1/2}}\right)=\sum_{j=1}^{N}p_{ij}\Phi\left(\frac{\ln\left(\kappa\frac{\lambda_{1z,i}}{\lambda_{1v,j}}\right)-\mu_{c,i}}{\omega_{c,i}^{1/2}}\right)\quad\text{if}\quad s_t=i.
\end{aligned}
$$

Finally, the denominator in (A.14) is given by:

$$
E\left[I_{\alpha,\kappa}\left(\frac{V_{t+1}}{\kappa\mathcal{R}_t\left(V_{t+1}\right)}\right)\mid J_t\right]=1+\left(\alpha^{-1}-1\right)\kappa^{1-\gamma}\sum_{j=1}^{N}p_{ij}\Phi\left(\frac{\ln\left(\kappa\frac{\lambda_{1z,i}}{\lambda_{1v,j}}\right)-\mu_{c,i}}{\omega_{c,i}^{1/2}}\right)\ \text{if}\ s_t=i. \tag{A.15}
$$

The numerator in (A.14) can be decomposed into two terms as follows:

$$
\begin{aligned}
& E\left[I_{\alpha,1}\left(\frac{V_{t+1}}{\kappa\mathcal{R}_t\left(V_{t+1}\right)}\right)\left(\lambda_{1v}^{\top}\zeta_{t+1}\right)^{1-\gamma}\exp\left(\left(1-\gamma\right)\Delta c_{t+1}\right)\mid J_t\right] \\
&\quad = E\left[\left(\lambda_{1v}^{\top}\zeta_{t+1}\right)^{1-\gamma}\exp\left(\left(1-\gamma\right)\Delta c_{t+1}\right)\mid J_t\right] \\
&\qquad + \left(\alpha^{-1}-1\right)E\left[\left(\lambda_{1v}^{\top}\zeta_{t+1}\right)^{1-\gamma}\exp\left(\left(1-\gamma\right)\Delta c_{t+1}\right)\mathbf{1}\left(\varepsilon_{c,t+1}<\frac{\ln\left(\kappa\frac{\lambda_{1z}^{\top}\zeta_t}{\lambda_{1v}^{\top}\zeta_{t+1}}\right)-\mu_c^{\top}\zeta_t}{\left(\omega_c^{\top}\zeta_t\right)^{1/2}}\right)\mid J_t\right]
\end{aligned}
$$

The first term is given by:

$$
\begin{aligned}
& E\left[\left(\lambda_{1v}^{\top}\zeta_{t+1}\right)^{1-\gamma}\exp\left(\left(1-\gamma\right)\Delta c_{t+1}\right)\mid J_t\right] \\
&\quad = E\left[\exp\left(\left(1-\gamma\right)\Delta c_{t+1}\right)\mid J_t\right]E\left[\left(\lambda_{1v}^{\top}\zeta_{t+1}\right)^{1-\gamma}\mid J_t\right] \\
&\quad = \exp\left(\left(1-\gamma\right)\mu_c^{\top}\zeta_t+\frac{\left(1-\gamma\right)^2}{2}\omega_c^{\top}\zeta_t\right)E\left[\left(\lambda_{1v}^{1-\gamma}\right)^{\top}\zeta_{t+1}\mid J_t\right] \\
&\quad = \exp\left(\left(1-\gamma\right)\mu_c^{\top}\zeta_t+\frac{\left(1-\gamma\right)^2}{2}\omega_c^{\top}\zeta_t\right)\left(\lambda_{1v}^{1-\gamma}\right)^{\top}P\zeta_t \\
&\quad = \exp\left(\left(1-\gamma\right)\mu_{c,i}+\frac{\left(1-\gamma\right)^2}{2}\omega_{c,i}\right)\sum_{j=1}^{N}p_{ij}\lambda_{1v,j}^{1-\gamma}\quad\text{if}\quad s_t=i,
\end{aligned}
\tag{A.16}
$$

where the first equality follows from that the processes ζ_{t+1} and Δc_{t+1} are independent conditional to the information J_t. In the second equality, we conveniently adopted the notation $a^q = (a_1^q, ..., a_N^q)^\top$ for $a \in \mathbb{R}_+^N$ and $q \in \mathbb{R}$. The expectation in the second term is given by:

$$
\begin{aligned}
&E\left[\left(\lambda_{1v}^\top \zeta_{t+1}\right)^{1-\gamma} \exp\left(\left(1-\gamma\right)\Delta c_{t+1}\right) \mathbf{1}\left(\varepsilon_{c,t+1} < \frac{\ln\left(\kappa \frac{\lambda_{1z}^\top \zeta_t}{\lambda_{1v}^\top \zeta_{t+1}}\right) - \mu_c^\top \zeta_t}{\left(\omega_c^\top \zeta_t\right)^{1/2}}\right) \mid J_t\right] \\
&\quad = E\Bigg[\left(\lambda_{1v}^\top \zeta_{t+1}\right)^{1-\gamma} E\Bigg[\exp\left(\left(1-\gamma\right)\Delta c_{t+1}\right) \\
&\qquad\qquad \times \mathbf{1}\left(\varepsilon_{c,t+1} < \frac{\ln\left(\kappa \frac{\lambda_{1z}^\top \zeta_t}{\lambda_{1v}^\top \zeta_{t+1}}\right) - \mu_c^\top \zeta_t}{\left(\omega_c^\top \zeta_t\right)^{1/2}}\right) \mid \{\zeta_\tau, \tau \in \mathbb{Z}\}, J_t\Bigg] \mid J_t\Bigg] \\
&\quad = E\Bigg[\left(\lambda_{1v}^\top \zeta_{t+1}\right)^{1-\gamma} \exp\left(\left(1-\gamma\right)\mu_c^\top \zeta_t + \frac{\left(1-\gamma\right)^2}{2}\omega_c^\top \zeta_t\right) \\
&\qquad\qquad \times \Phi\left(\frac{\ln\left(\kappa \frac{\lambda_{1z}^\top \zeta_t}{\lambda_{1v}^\top \zeta_{t+1}}\right) - \mu_c^\top \zeta_t}{\left(\omega_c^\top \zeta_t\right)^{1/2}} - \left(1-\gamma\right)\left(\omega_c^\top \zeta_t\right)^{1/2}\right) \mid J_t\Bigg] \\
&\quad = \exp\left(\left(1-\gamma\right)\mu_c^\top \zeta_t + \frac{\left(1-\gamma\right)^2}{2}\omega_c^\top \zeta_t\right) \\
&\qquad\qquad \times E\left[\left(\lambda_{1v}^\top \zeta_{t+1}\right)^{1-\gamma} \Phi\left(\frac{\ln\left(\kappa \frac{\lambda_{1z}^\top \zeta_t}{\lambda_{1v}^\top \zeta_{t+1}}\right) - \mu_c^\top \zeta_t}{\left(\omega_c^\top \zeta_t\right)^{1/2}} - \left(1-\gamma\right)\left(\omega_c^\top \zeta_t\right)^{1/2}\right) \mid s_t\right] \\
&\quad = \exp\left(\left(1-\gamma\right)\mu_{c,i} + \frac{\left(1-\gamma\right)^2}{2}\omega_{c,i}\right) \\
&\qquad\qquad \times \sum_{j=1}^N p_{ij} \lambda_{1v,j}^{1-\gamma} \Phi\left(\frac{\ln\left(\kappa \frac{\lambda_{1z,i}}{\lambda_{1v,j}}\right) - \mu_{c,i}}{\omega_{c,i}^{1/2}} - \left(1-\gamma\right)\omega_{c,i}^{1/2}\right) \quad \text{if} \quad s_t = i, \qquad \text{(A.17)}
\end{aligned}
$$

where the second equality follows from the property (A.11).

Finally, the numerator in (A.14), obtained by summing up (A.16) and (A.17), is given by:

$$
\begin{aligned}
&E\left[I_{\alpha,1}\left(\frac{V_{t+1}}{\kappa \mathcal{R}_t\left(V_{t+1}\right)}\right)\left(\lambda_{1v}^\top \zeta_{t+1}\right)^{1-\gamma} \exp\left(\left(1-\gamma\right)\Delta c_{t+1}\right) \mid J_t\right] \\
&\quad = \exp\left(\left(1-\gamma\right)\mu_{c,i} + \frac{\left(1-\gamma\right)^2}{2}\omega_{c,i}\right) \\
&\quad \times \sum_{j=1}^N p_{ij} \lambda_{1v,j}^{1-\gamma}\left(1 + \left(\alpha^{-1} - 1\right)\Phi\left(\frac{\ln\left(\kappa \frac{\lambda_{1z,i}}{\lambda_{1v,j}}\right) - \mu_{c,i}}{\omega_{c,i}^{1/2}} - \left(1-\gamma\right)\omega_{c,i}^{1/2}\right)\right) \quad \text{if} \quad s_t = i.
\end{aligned}
$$

dividing this expression by (A.15) and taking the power $1/(1-\gamma)$ gives the result:

$$\lambda_{1z,i} = \exp\left(\mu_{c,i} + \frac{1-\gamma}{2}\omega_{c,i}\right) \times \left(\sum_{j=1}^{N} p_{ij} \frac{1 + \left(\alpha^{-1} - 1\right)\Phi\left(\frac{\ln\left(\kappa\frac{\lambda_{1z,i}}{\lambda_{1v,j}}\right) - \mu_{c,i}}{\omega_{c,i}^{1/2}} - (1-\gamma)\,\omega_{c,i}^{1/2}\right)}{1 + \left(\alpha^{-1} - 1\right)\kappa^{1-\gamma}\sum_{j=1}^{N} p_{ij}\Phi\left(\frac{\ln\left(\kappa\frac{\lambda_{1z,i}}{\lambda_{1v,j}}\right) - \mu_{c,i}}{\omega_{c,i}^{1/2}}\right)} \lambda_{1v,j}^{1-\gamma} \right)^{\frac{1}{1-\gamma}} \tag{A.18}$$

and we define

$$p_{ij}^{*} = p_{ij} \frac{1 + \left(\alpha^{-1} - 1\right)\Phi\left(\frac{\ln\left(\kappa\frac{\lambda_{1z,i}}{\lambda_{1v,j}}\right) - \mu_{c,i}}{\omega_{c,i}^{1/2}} - (1-\gamma)\,\omega_{c,i}^{1/2}\right)}{1 + \left(\alpha^{-1} - 1\right)\kappa^{1-\gamma}\sum_{j=1}^{N} p_{ij}\Phi\left(\frac{\ln\left(\kappa\frac{\lambda_{1z,i}}{\lambda_{1v,j}}\right) - \mu_{c,i}}{\omega_{c,i}^{1/2}}\right)} \tag{A.19}$$

so that:

$$\lambda_{1z,i} = \exp\left(\mu_{c,i} + \frac{1-\gamma}{2}\omega_{c,i}\right)\left(\sum_{j=1}^{N} p_{ij}^{*}\lambda_{1v,j}^{1-\gamma}\right)^{\frac{1}{1-\gamma}}. \tag{A.20}$$

B.1.2 Second utility-consumption ratio

Dividing by C_t each side of the recursion

$$V_t = \left\{(1-\delta)\,C_t^{1-\frac{1}{\psi}} + \delta\left[\mathcal{R}_t\left(V_{t+1}\right)\right]^{1-\frac{1}{\psi}}\right\}^{\frac{1}{1-\frac{1}{\psi}}}, \tag{A.21}$$

it follows that

$$\frac{V_t}{C_t} = \left\{(1-\delta) + \delta\left[\frac{\mathcal{R}_t\left(V_{t+1}\right)}{C_t}\right]^{1-\frac{1}{\psi}}\right\}^{\frac{1}{1-\frac{1}{\psi}}} \tag{A.22}$$

or

$$\lambda_{1v}^{\top}\zeta_t = \left\{(1-\delta) + \delta\left(\lambda_{1z}^{\top}\zeta_t\right)^{1-\frac{1}{\psi}}\right\}^{\frac{1}{1-\frac{1}{\psi}}}, \tag{A.23}$$

and finally

$$\lambda_{1v,i} = \left\{(1-\delta) + \delta\lambda_{1z,i}^{1-\frac{1}{\psi}}\right\}^{\frac{1}{1-\frac{1}{\psi}}}, \quad \text{if} \quad s_t = i. \tag{A.24}$$

B.2 Price-Dividend Ratio

Given the stochastic discount factor

$$M_{t,t+1} = \delta \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\psi}} \left(\frac{V_{t+1}}{\mathcal{R}_t \left(V_{t+1} \right)} \right)^{\frac{1}{\psi} - \gamma} \frac{I_{\alpha,1} \left(\frac{V_{t+1}}{\kappa \mathcal{R}_t (V_{t+1})} \right)}{E \left[I_{\alpha,\kappa} \left(\frac{V_{t+1}}{\kappa \mathcal{R}_t (V_{t+1})} \right) \mid J_t \right]} \tag{A.25}$$

and the pricing equation

$$P_{d,t} = E \left[M_{t,t+1} \left(P_{t+1} + D_{t+1} \right) \mid J_t \right], \tag{A.26}$$

the price-dividend ratio is given by:

$$\begin{aligned}
\frac{P_{d,t}}{D_t} &= E \left[\delta \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\psi}} \left(\frac{V_{t+1}}{\mathcal{R}_t \left(V_{t+1} \right)} \right)^{\frac{1}{\psi} - \gamma} \frac{I_{\alpha,1} \left(\frac{V_{t+1}}{\kappa \mathcal{R}_t (V_{t+1})} \right)}{E \left[I_{\alpha,\kappa} \left(\frac{V_{t+1}}{\kappa \mathcal{R}_t (V_{t+1})} \right) \mid J_t \right]} \right. \\
&\qquad \left. \times \left(\frac{P_{d,t+1}}{D_{t+1}} + 1 \right) \frac{D_{t+1}}{D_t} \mid J_t \right] \\
&= E \left[\delta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \left(\frac{V_{t+1}/C_{t+1}}{\mathcal{R}_t \left(V_{t+1} \right) / C_t} \right)^{\frac{1}{\psi} - \gamma} \frac{I_{\alpha,1} \left(\frac{V_{t+1}}{\kappa \mathcal{R}_t (V_{t+1})} \right)}{E \left[I_{\alpha,\kappa} \left(\frac{V_{t+1}}{\kappa \mathcal{R}_t (V_{t+1})} \right) \mid J_t \right]} \right. \\
&\qquad \left. \times \left(\frac{P_{d,t+1}}{D_{t+1}} + 1 \right) \frac{D_{t+1}}{D_t} \mid J_t \right] \\
&= \delta E \left[\frac{I_{\alpha,1} \left(\frac{V_{t+1}}{\kappa \mathcal{R}_t (V_{t+1})} \right)}{E \left[I_{\alpha,\kappa} \left(\frac{V_{t+1}}{\kappa \mathcal{R}_t (V_{t+1})} \right) \mid J_t \right]} \left(\frac{\lambda_{1v}^\top \zeta_{t+1}}{\lambda_{1z}^\top \zeta_t} \right)^{\frac{1}{\psi} - \gamma} \right. \\
&\qquad \left. \times \left(\frac{P_{d,t+1}}{D_{t+1}} + 1 \right) \exp \left(-\gamma \Delta c_{t+1} + \Delta d_{t+1} \right) \mid J_t \right]
\end{aligned}$$

or

$$\lambda_{1d}^\top \zeta_t = \delta \frac{E \left[I_{\alpha,1} \left(\frac{V_{t+1}}{\kappa \mathcal{R}_t (V_{t+1})} \right) \left(\frac{\lambda_{1v}^\top \zeta_{t+1}}{\lambda_{1z}^\top \zeta_t} \right)^{\frac{1}{\psi} - \gamma} \left(\lambda_{1d}^\top \zeta_{t+1} + 1 \right) \exp \left(-\gamma \Delta c_{t+1} + \Delta d_{t+1} \right) \mid J_t \right]}{E \left[I_{\alpha,\kappa} \left(\frac{V_{t+1}}{\kappa \mathcal{R}_t (V_{t+1})} \right) \mid J_t \right]}. \tag{A.27}$$

Notice that $-\gamma \Delta c_{t+1} + \Delta d_{t+1} = \mu_{cd}^\top \zeta_t + \left(\omega_{cd}^\top \zeta_t \right)^{1/2} \varepsilon_{cd,t+1}$ where the new defined vectors are $\mu_{cd} = -\gamma \mu_c + \mu_d$ and $\omega_{cd} = \omega_c + \omega_d - 2\gamma \rho \odot \omega_c^{1/2} \odot \omega_d^{1/2}$.

As for the numerator in (A.14), the numerator in (A.27) can also be decomposed into two terms. The first term is:

$$
\begin{aligned}
&E\left[\left(\frac{\lambda_{1v}^\top\zeta_{t+1}}{\lambda_{1z}^\top\zeta_t}\right)^{\frac{1}{\psi}-\gamma}\left(\lambda_{1d}^\top\zeta_{t+1}+1\right)\exp\left(-\gamma\Delta c_{t+1}+\Delta d_{t+1}\right)\mid J_t\right]\\
&\quad= E\left[\exp\left(-\gamma\Delta c_{t+1}+\Delta d_{t+1}\right)\mid J_t\right]E\left[\left(\frac{\lambda_{1v}^\top\zeta_{t+1}}{\lambda_{1z}^\top\zeta_t}\right)^{\frac{1}{\psi}-\gamma}\left(\lambda_{1d}^\top\zeta_{t+1}+1\right)\mid J_t\right]\\
&\quad= \exp\left(\mu_{cd}^\top\zeta_t+\frac{1}{2}\omega_{cd}^\top\zeta_t\right)\frac{\left(\lambda_{1v}^{1-1/\psi}\odot\left(\lambda_{1d}+e\right)\right)^\top P\zeta_t}{\left(\lambda_{1z}^{1-1/\psi}\right)^\top\zeta_t}\\
&\quad= \left(\frac{1}{\lambda_{1z,i}}\right)^{1-1/\psi}\exp\left(\mu_{cd,i}+\frac{1}{2}\omega_{cd,i}\right)\sum_{j=1}^{N}p_{ij}\lambda_{1v,j}^{1-1/\psi}\left(\lambda_{1d,j}+1\right)\quad\text{if}\quad s_t=i,
\end{aligned}
\tag{A.28}
$$

since ζ_{t+1} and $-\gamma\Delta c_{t+1}+\Delta d_{t+1}$ are independent, given the information J_t, and $1=e^\top\zeta_{t+1}$.

The second term is up to the multiplicative constant $(\alpha^{-1}-1)$, given by:

$$
\begin{aligned}
&E\Bigg[\left(\frac{\lambda_{1v}^\top\zeta_{t+1}}{\lambda_{1z}^\top\zeta_t}\right)^{\frac{1}{\psi}-\gamma}\left(\lambda_{1d}^\top\zeta_{t+1}+1\right)\exp\left(-\gamma\Delta c_{t+1}+\Delta d_{t+1}\right)\\
&\qquad\qquad\qquad\times\mathbf{1}\left(\varepsilon_{c,t+1}<\frac{\ln\left(\kappa\frac{\lambda_{1z}^\top\zeta_t}{\lambda_{1v}^\top\zeta_{t+1}}\right)-\mu_c^\top\zeta_t}{\left(\omega_c^\top\zeta_t\right)^{1/2}}\right)\mid J_t\Bigg]\\
&= E\Bigg[\left(\frac{\lambda_{1v}^\top\zeta_{t+1}}{\lambda_{1z}^\top\zeta_t}\right)^{\frac{1}{\psi}-\gamma}\left(\lambda_{1d}^\top\zeta_{t+1}+1\right)\exp\left(\mu_{cd}^\top\zeta_t+\frac{1}{2}\omega_{cd}^\top\zeta_t\right)\\
&\qquad\times\Phi\left(\frac{\ln\left(\kappa\frac{\lambda_{1z}^\top\zeta_t}{\lambda_{1v}^\top\zeta_{t+1}}\right)-\mu_c^\top\zeta_t}{\left(\omega_c^\top\zeta_t\right)^{1/2}}-\left(\left(\rho^\top\zeta_t\right)\left(\omega_d^\top\zeta_t\right)^{1/2}-\gamma\left(\omega_c^\top\zeta_t\right)^{1/2}\right)\right)\mid J_t\Bigg]\\
&= \left(\frac{1}{\lambda_{1z,i}}\right)^{1-1/\psi}\exp\left(\mu_{cd,i}+\frac{1}{2}\omega_{cd,i}\right)\\
&\qquad\times\sum_{j=1}^{N}p_{ij}\lambda_{1v,j}^{1-1/\psi}\left(\lambda_{1d,j}+1\right)\Phi\left(\frac{\ln\left(\kappa\frac{\lambda_{1z,i}}{\lambda_{1v,j}}\right)-\mu_{c,i}}{\omega_{c,i}^{1/2}}-\left(\rho_i\omega_{d,i}^{1/2}-\gamma\omega_{c,i}^{1/2}\right)\right)\quad\text{if}\quad s_t=i,
\end{aligned}
\tag{A.29}
$$

where we first condition on $\langle\{\zeta_\tau,\tau\in\mathbb{Z}\},J_t\rangle$ by law of iterated expectations and use (A.11) for the expectation conditional on $\langle\{\zeta_\tau,\tau\in\mathbb{Z}\},J_t\rangle$. The denominator in (A.27) is already computed and given by (A.15). Summing up (A.28) and (A.29) and dividing by (A.15), (A.27) becomes:

$$\lambda_{1d,i} = \delta \left(\frac{1}{\lambda_{1z,i}} \right)^{1-1/\psi} \exp \left(\mu_{cd,i} + \frac{1}{2} \omega_{cd,i} \right)$$
$$\times \sum_{j=1}^{N} p_{ij} \frac{1 + \left(\alpha^{-1} - 1 \right) \Phi \left(\frac{\ln \left(\kappa \frac{\lambda_{1z,i}}{\lambda_{1v,j}} \right) - \mu_{c,i}}{\omega_{c,i}^{1/2}} - \left(\rho_i \omega_{d,i}^{1/2} - \gamma \omega_{c,i}^{1/2} \right) \right)}{1 + \left(\alpha^{-1} - 1 \right) \kappa^{1-\gamma} \sum_{j=1}^{N} p_{ij} \Phi \left(\frac{\ln \left(\kappa \frac{\lambda_{1z,i}}{\lambda_{1v,j}} \right) - \mu_{c,i}}{\omega_{c,i}^{1/2}} \right)} \lambda_{1v,j}^{1-1/\psi} \left(\lambda_{1d,j} + 1 \right) \tag{A.30}$$

and we set

$$p_{ij}^{**} = p_{ij} \frac{1 + \left(\alpha^{-1} - 1 \right) \Phi \left(\frac{\ln \left(\kappa \frac{\lambda_{1z,i}}{\lambda_{1v,j}} \right) - \mu_{c,i}}{\omega_{c,i}^{1/2}} - \left(\rho_i \omega_{d,i}^{1/2} - \gamma \omega_{c,i}^{1/2} \right) \right)}{1 + \left(\alpha^{-1} - 1 \right) \kappa^{1-\gamma} \sum_{j=1}^{N} p_{ij} \Phi \left(\frac{\ln \left(\kappa \frac{\lambda_{1z,i}}{\lambda_{1v,j}} \right) - \mu_{c,i}}{\omega_{c,i}^{1/2}} \right)} \tag{A.31}$$

so that:

$$\lambda_{1d,i} = \delta \left(\frac{1}{\lambda_{1z,i}} \right)^{1-1/\psi} \exp \left(\mu_{cd,i} + \frac{1}{2} \omega_{cd,i} \right) \sum_{j=1}^{N} p_{ij}^{**} \lambda_{1v,j}^{1-1/\psi} \left(\lambda_{1d,j} + 1 \right). \tag{A.32}$$

We also have the following Lemma.
Lemma 1: *The solution to the linear system:*

$$u_i = v_i \sum_{j=1}^{N} p_{ij} w_j \left(1 + u_j \right) \;\; \forall i = 1, .., N$$

with unknowns $u_i, \;\; i = 1, .., N$ *is given by:*

$$u_i = v_i w^{\top} P \left[Id - D_{vw} P \right]^{-1} e_i \tag{A.33}$$

where e_i *is the* $N \times 1$ *vector of zeroes but one at the position* i, $u = \left(u_1, .., u_N \right)^{\top}$, $v = \left(v_1, .., v_N \right)^{\top}$, $w = \left(w_1, .., w_N \right)^{\top}$ *and* D_{vw} *is the diagonal matrix* $D_{vw} = Diag \left(v_1 w_1, .., v_N w_N \right)$.

We use Lemma 1 to write the solution to the linear system (A.32) as:

$$\lambda_{1d,i} = \delta \left(\frac{1}{\lambda_{1z,i}} \right)^{\frac{1}{\psi} - \gamma} \exp \left(\mu_{cd,i} + \frac{\omega_{cd,i}}{2} \right) \left(\lambda_{1v}^{\frac{1}{\psi} - \gamma} \right)^{\top} P^{**} \left(Id - \delta A^{**} \left(\mu_{cd} + \frac{\omega_{cd}}{2} \right) \right)^{-1} e_i$$

where

$$A^{**} \left(u \right) = Diag \left(\left(\frac{\lambda_{1v,1}}{\lambda_{1z,1}} \right)^{\frac{1}{\psi} - \gamma} \exp \left(u_1 \right), ..., \left(\frac{\lambda_{1v,N}}{\lambda_{1z,N}} \right)^{\frac{1}{\psi} - \gamma} \exp \left(u_N \right) \right) P^{**}$$

and

$$P^{**\top} = \left[p_{ij}^{**} \right]_{1 \leq i,j \leq N}.$$

C Proofs of Formulas for Reported Statistics

We have the following Lemma.
Lemma 2: *For any vectors $a,\ b \in \mathbb{R}^N$ and for any integer h, $h > 0$, we have*

$$\begin{aligned} & Var\left[\sum_{j=1}^{h}\left(a^{\top}\zeta_{t+j-1}\right)\left(b^{\top}\zeta_{t+j}\right)\right] \\ & \quad = h\left(a\odot a\right)^{\top}E\left[\zeta_t\zeta_t^{\top}\right]P^{\top}\left(b\odot b\right) - h^2\left(a^{\top}E\left[\zeta_t\zeta_t^{\top}\right]P^{\top}b\right)^2 \\ & \qquad + 2\sum_{j=2}^{h}\left(h-j+1\right)a^{\top}E\left[\zeta_t\zeta_t^{\top}\right]P^{\top}\left(b\odot\left(\left(P^{j-2}\right)^{\top}\left(a\odot\left(P^{\top}b\right)\right)\right)\right). \end{aligned} \tag{A.34}$$

Proof of Lemma 2. Define the random variable u_t as $u_t = \left(a^{\top}\zeta_{t-1}\right)\left(b^{\top}\zeta_t\right)$. We have

$$\begin{aligned} Var\left[\sum_{j=1}^{h}\left(a^{\top}\zeta_{t+j-1}\right)\left(b^{\top}\zeta_{t+j}\right)\right] &= Var\left[\sum_{j=1}^{h}u_{t+j}\right] \\ &= hVar\left[u_t\right] + 2\sum_{j=2}^{h}\left(h-j+1\right)Cov\left(u_{t+1},u_{t+j}\right). \end{aligned} \tag{A.35}$$

We first compute $Var\left[u_t\right]$. We have,

$$E\left[u_t\right] = a^{\top}E\left[\zeta_t\zeta_{t+1}^{\top}\right]b = a^{\top}E\left[\zeta_t E\left[\zeta_{t+1}^{\top}\mid\zeta_t\right]\right]b = a^{\top}E\left[\zeta_t\zeta_t^{\top}P^{\top}\right]b = a^{\top}E\left[\zeta_t\zeta_t^{\top}\right]P^{\top}b. \tag{A.36}$$

In addition,

$$u_t^2 = \left(a^{\top}\zeta_t\right)^2\left(b^{\top}\zeta_{t+1}\right)^2 = \left(\left(a\odot a\right)^{\top}\zeta_t\right)\left(\left(b\odot b\right)^{\top}\zeta_{t+1}\right).$$

Hence, the same calculations done in the proof of (A.36) yields to

$$E\left[u_t^2\right] = \left(a\odot a\right)^{\top}E\left[\zeta_t\zeta_t^{\top}\right]P^{\top}\left(b\odot b\right). \tag{A.37}$$

By combining (A.36) and (A.37), one gets

$$Var\left[u_t\right] = \left(a\odot a\right)^{\top}E\left[\zeta_t\zeta_t^{\top}\right]P^{\top}\left(b\odot b\right) - \left(a^{\top}E\left[\zeta_t\zeta_t^{\top}\right]P^{\top}b\right)^2. \tag{A.38}$$

We now compute $Cov\left(u_{t+1},u_{t+j}\right)$. For $j \geq 2$, we have

$$\begin{aligned} E\left[u_{t+1}u_{t+j}\right] &= E\left[\left(a^{\top}\zeta_t\right)\left(b^{\top}\zeta_{t+1}\right)\left(a^{\top}\zeta_{t+j-1}\right)\left(b^{\top}\zeta_{t+j}\right)\right] \\ &= E\left[\left(a^{\top}\zeta_t\right)\left(b^{\top}\zeta_{t+1}\right)\left(a\top\zeta_{t+j-1}\right)\left(b^{\top}E\left[\zeta_{t+j}\mid\zeta_{t+j-1}\right]\right)\right] \\ &= E\left[\left(a^{\top}\zeta_t\right)\left(b^{\top}\zeta_{t+1}\right)\left(a^{\top}\zeta_{t+j-1}\right)\left(b^{\top}P\zeta_{t+j-1}\right)\right] \\ &= E\left[\left(a^{\top}\zeta_t\right)\left(b^{\top}\zeta_{t+1}\right)\left(\left(a\odot\left(P^{\top}b\right)\right)^{\top}\zeta_{t+j-1}\right)\right], \end{aligned}$$

where the last equality follows from (A.10). Hence,

$$\begin{aligned}
E\left[u_{t+1}u_{t+j}\right] &= E\left[\left(a^\top\zeta_t\right)\left(b^\top\zeta_{t+1}\right)\left(\left(a\odot\left(P^\top b\right)\right)^\top E\left[\zeta_{t+j-1}\mid\zeta_{t+1}\right]\right)\right]\\
&= E\left[\left(a^\top\zeta_t\right)\left(b^\top\zeta_{t+1}\right)\left(\left(a\odot\left(P^\top b\right)\right)^\top P^{j-2}\zeta_{t+1}\right)\right]\\
&= E\left[\left(a^\top\zeta_t\right)\left(b\odot\left(\left(P^{j-2}\right)^\top\left(a\odot\left(P^\top b\right)\right)\right)^\top\zeta_{t+1}\right)\right],
\end{aligned}$$

where again the last equality follows from (A.10). Therefore,

$$\begin{aligned}
E\left[u_{t+1}u_{t+j}\right] &= a^\top E\left[\zeta_t\zeta_{t+1}^\top\right]\left(b\odot\left(\left(P^{j-2}\right)^\top\left(a\odot\left(P^\top b\right)\right)\right)\right)\\
&= a^\top E\left[\zeta_t\zeta_t^\top\right]P^\top\left(b\odot\left(\left(P^{j-2}\right)^\top\left(a\odot\left(P^\top b\right)\right)\right)\right).
\end{aligned} \tag{A.39}$$

By combining (A.36) and (A.39), one gets

$$Cov\left(u_{t+1},u_{t+j}\right) = a^\top E\left[\zeta_t\zeta_t^\top\right]P^\top\left(b\odot\left(\left(P^{j-2}\right)^\top\left(a\odot\left(P^\top b\right)\right)\right)\right) - \left(a^\top E\left[\zeta_t\zeta_t^\top\right]P^\top b\right)^2. \tag{A.40}$$

By plugging (A.38) and (A.40) into (A.35), one gets (A.34). We also have the following Lemma.

Lemma 3: *For any vectors $a,\ b,\ c,\ d\in\mathbb{R}^N$ and for any integer h, $h>0$, we have*

$$\begin{aligned}
&Cov\left(\sum_{j=1}^{h}\left(a^\top\zeta_{t+j-1}\right)\left(b^\top\zeta_{t+j}\right),\sum_{j=1}^{h}\left(c^\top\zeta_{t+j-1}\right)\left(d^\top\zeta_{t+j}\right)\right)\\
&\quad= \sum_{j=1}^{h}\left(a\odot c\right)^\top E\left[\zeta_t\zeta_t^\top\right]P^\top\left(b\odot d\right) - h^2\left(a^\top E\left[\zeta_t\zeta_t^\top\right]P^\top b\right)\left(c^\top E\left[\zeta_t\zeta_t^\top\right]P^\top d\right)\\
&\qquad+ \sum_{j=2}^{h}a^\top E\left[\zeta_t\zeta_t^\top\right]P^\top\left(b\odot\left(\left(\sum_{i=0}^{j-2}P^i\right)^\top\left(c\odot\left(P^\top d\right)\right)\right)\right)\\
&\qquad+ \sum_{j=2}^{h}c^\top E\left[\zeta_t\zeta_t^\top\right]P^\top\left(d\odot\left(\left(\sum_{i=0}^{j-2}P^i\right)^\top\left(a\odot\left(P^\top b\right)\right)\right)\right).
\end{aligned} \tag{A.41}$$

Proof of Lemma 3. Similar techniques and hints are used as for the proof of Lemma 2. Lemma 2 is also a particular case of Lemma 3.

C.1 Expected Values

We have

$$\begin{aligned}
R_{t+1} &= \frac{P_{d,t+1}+D_{t+1}}{P_{d,t}} = \frac{D_t}{P_{d,t}}\frac{D_{t+1}}{D_t}\left(\frac{P_{d,t+1}}{D_{t+1}}+1\right) = \left(\lambda_{2d}^\top\zeta_t\right)\exp\left(\Delta d_{t+1}\right)\left(\lambda_{1d}^\top\zeta_{t+1}+1\right)\\
&= \left(\lambda_{2d}^\top\zeta_t\right)\exp\left(\Delta d_{t+1}\right)\left(\lambda_{3d}^\top\zeta_{t+1}\right),
\end{aligned}$$

where the last equality holds given that $e^\top \zeta_{t+1} = 1$. Given the information J_t, the processes ζ_{t+1} and Δd_{t+1} are independent. Therefore,

$$\begin{aligned}
E\left[R_{t+1} \mid J_t\right] &= E\left[\left(\lambda_{2d}^\top \zeta_t\right) \exp\left(\Delta d_{t+1}\right) \left(\lambda_{3d}^\top \zeta_{t+1}\right) \mid J_t\right] \\
&= \left(\lambda_{2d}^\top \zeta_t\right) E\left[\exp\left(\Delta d_{t+1}\right) \mid J_t\right] E\left[\left(\lambda_{3d}^\top \zeta_{t+1}\right) \mid J_t\right] \\
&= \left(\lambda_{2d}^\top \zeta_t\right) \exp\left(\mu_d^\top \zeta_t + \omega_d^\top \zeta_t / 2\right) \lambda_{3d}^\top E\left[\zeta_{t+1} \mid J_t\right] \\
&= \left(\lambda_{2d}^\top \zeta_t\right) \exp\left(\mu_d^\top \zeta_t + \omega_d^\top \zeta_t / 2\right) \lambda_{3d}^\top P \zeta_t \\
&= \psi_d^\top \zeta_t.
\end{aligned}$$

Consequently, $\forall j \geq 2$

$$E\left[R_{t+j} \mid J_t\right] = \psi_d^\top E\left[\zeta_{t+j-1} \mid J_t\right] = \psi_d^\top P^{j-1} \zeta_t.$$

Finally,

$$E\left[R_{t+1:t+h} \mid J_t\right] = E\left[\sum_{j=1}^{h} R_{t+j} \mid J_t\right] = \psi_d^\top \left(\sum_{j=1}^{h} P^{j-1}\right) \zeta_t = \psi_{h,d}^\top \zeta_t.$$

Aggregate consumption and dividend growth rates over h periods are defined by:

$$\Delta c_{t+1:t+h} = \sum_{j=1}^{h} \Delta c_{t+j} \text{ and } \Delta d_{t+1:t+h} = \sum_{j=1}^{h} \Delta d_{t+j}.$$

Similar arguments and techniques can be used to prove that the expected values of these multi-period growth rates are given by:

$$E\left[\Delta c_{t+1:t+h} \mid J_t\right] = \mu_{ch}^\top \zeta_t \text{ and } E\left[\Delta d_{t+1:t+h} \mid J_t\right] = \mu_{dh}^\top \zeta_t$$

where

$$\mu_{ch} = \left(\sum_{j=1}^{h} P^{j-1}\right)^\top \mu_c \text{ and } \mu_{dh} = \left(\sum_{j=1}^{h} P^{j-1}\right)^\top \mu_d.$$

C.2 Covariances

We also have

$$\begin{aligned}
Cov\left(R_{t+1:t+h}, \frac{D_t}{P_t}\right) &= Cov\left(E\left[R_{t+1:t+h} \mid J_t\right], \lambda_{2d}^\top \zeta_t\right) = Cov\left(\psi_{h,d}^\top \zeta_t, \lambda_{2d}^\top \zeta_t\right) \\
&= \psi_{h,d}^\top Cov\left(\zeta_t, \zeta_t^\top \lambda_{2d}\right) = \psi_{h,d}^\top Var\left[\zeta_t\right] \lambda_{2d}.
\end{aligned}$$

Similar arguments and techniques are used to prove that covariances of growth rates with the dividend-price ratio are given by:

$$Cov\left(\Delta c_{t+1:t+h}, \frac{D_t}{P_t}\right) = \mu_{ch}^\top Var\left[\zeta_t\right] \lambda_{2d} \tag{A.42}$$

$$Cov\left(\Delta d_{t+1:t+h}, \frac{D_t}{P_t}\right) = \mu_{dh}^\top Var\left[\zeta_t\right] \lambda_{2d}. \tag{A.43}$$

C.3 Variances

Observe that conditional on the information set $\{\zeta_\tau, \tau \in \mathbb{Z}\}$, the variables R_{t+j}, $j = 1, ..., h$, are independent. Therefore,

$$\begin{aligned} Var\left[R_{t+1:t+h}\right] &= Var\left[E\left[R_{t+1:t+h} \mid \{\zeta_\tau, \tau \in \mathbb{Z}\}\right]\right] + E\left[Var\left[R_{t+1:t+h} \mid \{\zeta_\tau, \tau \in \mathbb{Z}\}\right]\right] \\ &= Var\left[\sum_{j=1}^{h} E\left[R_{t+j} \mid \{\zeta_\tau, \tau \in \mathbb{Z}\}\right]\right] + E\left[\sum_{j=1}^{h} Var\left[R_{t+j} \mid \{\zeta_\tau, \tau \in \mathbb{Z}\}\right]\right]. \end{aligned} \tag{A.44}$$

Given that $R_{t+j} = \left(\lambda_{2d}^\top \zeta_{t+j-1}\right)\left(\lambda_{3d}^\top \zeta_{t+j}\right)\exp\left(\Delta d_{t+j}\right)$, we have

$$\begin{aligned} E\left[R_{t+j} \mid \{\zeta_\tau, \tau \in \mathbb{Z}\}\right] &= \left(\lambda_{2d}^\top \zeta_{t+j-1}\right)\left(\lambda_{3d}^\top \zeta_{t+j}\right) E\left[\exp\left(\Delta d_{t+j}\right) \mid \{\zeta_\tau, \tau \in \mathbb{Z}\}\right] \\ &= \left(\lambda_{2d}^\top \zeta_{t+j-1}\right)\left(\lambda_{3d}^\top \zeta_{t+j}\right)\exp\left(\mu_d^\top \zeta_{t+j-1} + \omega_d^\top \zeta_{t+j-1}/2\right) \\ &= \left(\theta_{1d}^\top \zeta_{t+j-1}\right)\left(\lambda_{3d}^\top \zeta_{t+j}\right), \end{aligned} \tag{A.45}$$

and

$$\begin{aligned} Var\left[R_{t+j} \mid \{\zeta_\tau, \tau \in \mathbb{Z}\}\right] &= \left(\lambda_{2d}^\top \zeta_{t+j-1}\right)^2\left(\lambda_{3d}^\top \zeta_{t+j}\right)^2 Var\left[\exp\left(\Delta d_{t+j}\right) \mid \{\zeta_\tau, \tau \in \mathbb{Z}\}\right] \\ &= \left(\left(\lambda_{2d} \odot \lambda_{2d}\right)^\top \zeta_{t+j-1}\right)\left(\left(\lambda_{3d} \odot \lambda_{3d}\right)^\top \zeta_{t+j}\right) \\ &\quad \left(\exp\left(2\mu_d^\top \zeta_{t+j-1} + 2\omega_d^\top \zeta_{t+j-1}\right) - \exp\left(2\mu_d^\top \zeta_{t+j-1} + \omega_d^\top \zeta_{t+j-1}\right)\right) \\ &= \left(\theta_{2d}^\top \zeta_{t+j-1}\right)\left(\theta_{3d}^\top \zeta_{t+j}\right). \end{aligned} \tag{A.46}$$

Consequently,

$$\begin{aligned} E\left[\sum_{j=1}^{h} Var\left[R_{t+j} \mid \{\zeta_\tau, \tau \in \mathbb{Z}\}\right]\right] &= E\left[\sum_{j=1}^{h}\left(\theta_{2d}^\top \zeta_{t+j-1}\right)\left(\theta_{3d}^\top \zeta_{t+j}\right)\right] \\ &= \theta_{2d}^\top \sum_{j=1}^{h} E\left[\zeta_{t+j-1}\zeta_{t+j}^\top\right]\theta_{3d} \\ &= \theta_{2d}^\top \sum_{j=1}^{h} E\left[\zeta_{t+j-1} E[\zeta_{t+j}^\top \mid J_{t+j-1}]\right]\theta_{3d} \\ &= \theta_{2d}^\top \sum_{j=1}^{h} E\left[\zeta_{t+j-1}\zeta_{t+j-1}^\top P^\top\right]\theta_{3d}, \end{aligned}$$

i.e.,

$$E\left[\sum_{j=1}^{h} Var\left[R_{t+j} \mid \{\zeta_\tau, \tau \in \mathbb{Z}\}\right]\right] = h\theta_{2d}^\top E\left[\zeta_t \zeta_t^\top\right] P^\top \theta_{3d}. \tag{A.47}$$

In addition, we have

$$Var\left[\sum_{j=1}^{h} E\left[R_{t+j} \mid \{\zeta_\tau, \tau \in \mathbb{Z}\}\right]\right] = Var\left[\sum_{j=1}^{h} \left(\theta_{1d}^\top \zeta_{t+j-1}\right)\left(\lambda_{3d}^\top \zeta_{t+j}\right)\right].$$

Therefore, by using (A.34), one gets

$$\begin{aligned} &Var\left[\sum_{j=1}^{h} E\left[R_{t+j} \mid \{\zeta_\tau, \tau \in \mathbb{Z}\}\right]\right] \\ &\quad = h\left(\theta_{1d} \odot \theta_{1d}\right)^\top E\left[\zeta_t \zeta_t^\top\right] P^\top \left(\lambda_{3d} \odot \lambda_{3d}\right) - h^2 \left(\theta_{1d}^\top E\left[\zeta_t \zeta_t^\top\right] P^\top \lambda_{3d}\right)^2 \\ &\qquad + 2\sum_{j=2}^{h} (h-j+1)\, \theta_{1d}^\top E\left[\zeta_t \zeta_t^\top\right] P^\top \left(\lambda_{3d} \odot \left(\left(P^{j-2}\right)^\top \left(\theta_{1d} \odot \left(P^\top \lambda_{3d}\right)\right)\right)\right). \end{aligned} \tag{A.48}$$

Finally, by combining (A.44) with (A.47) and (A.48), one gets the variance of aggregate returns:

$$\begin{aligned} Var\left[R_{t+1:t+h}\right] = {} & h\theta_{2d}^\top E\left[\zeta_t \zeta_t^\top\right] P^\top \theta_{3d}. \\ & + h\left(\theta_{1d} \odot \theta_{1d}\right)^\top E\left[\zeta_t \zeta_t^\top\right] P^\top \left(\lambda_{3d} \odot \lambda_{3d}\right) - h^2 \left(\theta_{1d}^\top E\left[\zeta_t \zeta_t^\top\right] P^\top \lambda_{3d}\right)^2 \\ & + 2\sum_{j=2}^{h} (h-j+1)\, \theta_{1d}^\top E\left[\zeta_t \zeta_t^\top\right] P^\top \left(\lambda_{3d} \odot \left(\left(P^{j-2}\right)^\top \left(\theta_{1d} \odot \left(P^\top \lambda_{3d}\right)\right)\right)\right), \end{aligned} \tag{A.49}$$

One has:

$$Var\left[R_{f,t+1:t+h}\right] = Var\left[\sum_{j=1}^{h} \left(\lambda_{2f}^\top \zeta_{t+j-1}\right)\right] = Var\left[\sum_{j=1}^{h} \left(\lambda_{2f}^\top \zeta_{t+j-1}\right)\left(e^\top \zeta_{t+j}\right)\right]$$

which can be computed directly from (A.34):

$$\begin{aligned} Var\left[R_{f,t+1:t+h}\right] = {} & h\left(\lambda_{2f} \odot \lambda_{2f}\right)^\top E\left[\zeta_t \zeta_t^\top\right] P^\top (e \odot e) - h^2 \left(\lambda_{2f}^\top E\left[\zeta_t \zeta_t^\top\right] P^\top e\right)^2 \\ & + 2\sum_{j=2}^{h} (h-j+1)\, \lambda_{2f}^\top E\left[\zeta_t \zeta_t^\top\right] P^\top \left(e \odot \left(\left(P^{j-2}\right)^\top \left(\lambda_{2f} \odot \left(P^\top e\right)\right)\right)\right). \end{aligned} \tag{A.50}$$

Also, one has:

$$\begin{aligned} Cov\left(R_{t+1:t+h}, R_{f,t+1:t+h}\right) &= Cov\left(E\left[R_{t+1:t+h} \mid \{\zeta_\tau, \tau \in \mathbb{Z}\}\right], R_{f,t+1:t+h}\right) \\ &= Cov\left(\sum_{j=1}^{h} \left(\theta_{1d}^\top \zeta_{t+j-1}\right)\left(\lambda_{3d}^\top \zeta_{t+j}\right), \sum_{j=1}^{h} \left(\lambda_{2f}^\top \zeta_{t+j-1}\right)\left(e^\top \zeta_{t+j}\right)\right) \end{aligned}$$

which can be computed directly from (A.41):

$$
\begin{aligned}
&Cov\left(R_{t+1:t+h}, R_{f,t+1:t+h}\right)\\
&\quad = \sum_{j=1}^{h} \left(\theta_{1d} \odot \lambda_{2f}\right)^{\top} E\left[\zeta_t \zeta_t^{\top}\right] P^{\top} \left(\lambda_{3d} \odot e\right) - h^2 \left(\theta_{1d}^{\top} E\left[\zeta_t \zeta_t^{\top}\right] P^{\top} \lambda_{3d}\right) \left(\lambda_{2f}^{\top} E\left[\zeta_t \zeta_t^{\top}\right] P^{\top} e\right)\\
&\qquad\qquad + \sum_{j=2}^{h} \theta_{1d}^{\top} E\left[\zeta_t \zeta_t^{\top}\right] P^{\top} \left(\lambda_{3d} \odot \left(\left(\sum_{i=0}^{j-2} P^i\right)^{\top} \left(\lambda_{2f} \odot \left(P^{\top} e\right)\right)\right)\right)\\
&\qquad\qquad + \sum_{j=2}^{h} \lambda_{2f}^{\top} E\left[\zeta_t \zeta_t^{\top}\right] P^{\top} \left(e \odot \left(\left(\sum_{i=0}^{j-2} P^i\right)^{\top} \left(\theta_{1d} \odot \left(P^{\top} \lambda_{3d}\right)\right)\right)\right).
\end{aligned}
\tag{A.51}
$$

Observe that the variance of aggregate excess returns is given by:

$$
Var\left[R^e_{t+1:t+h}\right] = Var\left[R_{t+1:t+h}\right] - 2Cov\left(R_{t+1:t+h}, R_{f,t+1:t+h}\right) + Var\left[R_{f,t+1:t+h}\right]
$$

which the formula is obtained by combining (A.49), (A.50) and (A.51).

Remark that:

$$
\begin{aligned}
Var\left[\zeta_{t+1:t+h}\right] = Var\left[\zeta_{t:t+h-1}\right] &= Var\left[\sum_{j=1}^{h} \zeta_{t+j-1}\right]\\
&= hVar\left[\zeta_t\right] + 2\sum_{j=2}^{h} (h-j+1)\, Cov\left(\zeta_t, \zeta_{t+j-1}\right)\\
&= hVar\left[\zeta_t\right] + 2\sum_{j=2}^{h} (h-j+1)\, Cov\left(\zeta_t, E\left[\zeta_{t+j-1} \mid J_t\right]\right)\\
&= hVar\left[\zeta_t\right] + 2\sum_{j=2}^{h} (h-j+1)\, Cov\left(\zeta_t, P^{j-1}\zeta_t\right)\\
&= \left(hI + 2\sum_{j=2}^{h} (h-j+1)\, P^{j-1}\right) Var\left[\zeta_t\right].
\end{aligned}
\tag{A.52}
$$

In addition, variances of growth rates are also given by:

$$
\begin{aligned}
Var\left[\Delta c_{t+1:t+h}\right] &= Var\left[E\left[\Delta c_{t+1:t+h} \mid \{\zeta_\tau, \tau \in \mathbb{Z}\}\right]\right] + E\left[Var\left[\Delta c_{t+1:t+h} \mid \{\zeta_\tau, \tau \in \mathbb{Z}\}\right]\right]\\
&= Var\left[\mu_c^{\top} \zeta_{t:t+h-1}\right] + E\left[\omega_c^{\top} \zeta_{t:t+h-1}\right]\\
&= \mu_c^{\top} Var\left[\zeta_{t:t+h-1}\right] \mu_c + h\omega_c^{\top} \Pi \qquad\qquad (A.53)\\
Var\left[\Delta d_{t+1:t+h}\right] &= Var\left[E\left[\Delta d_{t+1:t+h} \mid \{\zeta_\tau, \tau \in \mathbb{Z}\}\right]\right] + E\left[Var\left[\Delta d_{t+1:t+h} \mid \{\zeta_\tau, \tau \in \mathbb{Z}\}\right]\right]\\
&= Var\left[\mu_d^{\top} \zeta_{t:t+h-1}\right] + E\left[\omega_d^{\top} \zeta_{t:t+h-1}\right]\\
&= \mu_d^{\top} Var\left[\zeta_{t:t+h-1}\right] \mu_d + h\omega_d^{\top} \Pi \qquad\qquad (A.54)
\end{aligned}
$$

where $Var\left[\zeta_{t:t+h-1}\right] = Var\left[\zeta_{t+1:t+h}\right]$ given by (A.52) and since $E\left[\zeta_{t:t+h-1}\right] = hE\left[\zeta_t\right] = h\Pi$.

C.4 Predictability of Volatility

To assess the predictive power of equity return volatility by the dividend price ratio, we first compute the population regression

$$R_{t+1} = a_R + b_R R_t + U_{R,t+1} \tag{A.55}$$

where R_{t+1} denotes the gross return on equity:

$$R_{t+1} = \left(\lambda_{2d}^\top \zeta_t\right) \exp\left(\mu_d^\top \zeta_t + \left(\omega_d^\top \zeta_t\right)^{1/2} \varepsilon_{d,t+1}\right) \left(\lambda_{3d}^\top \zeta_{t+1}\right). \tag{A.56}$$

It follows that

$$U_{R,t+1} = R_{t+1} - b_R R_t - a_R \quad \text{where} \quad b_R = \frac{Cov\left(R_{t+1}, R_t\right)}{Var\left[R_t\right]} \quad \text{and} \quad a_R = E\left[R_{t+1}\right]\left(1 - b_R\right). \tag{A.57}$$

We measure volatility as a moving sum of these squared residuals and consider the predictive regression

$$\sum_{j=1}^{h} U_{R,t+j}^2 = a\left(h\right) + b\left(h\right)\frac{D_t}{P_t} + \eta_{t+h}\left(h\right). \tag{A.58}$$

Notice that

$$U_{R,t+j}^2 = R_{t+j}^2 + b_R^2 R_{t+j-1}^2 - 2b_R R_{t+j} R_{t+j-1} - 2a_R R_{t+j} + 2a_R b_R R_{t+j-1} + a_R^2, \;\; j \geq 1. \tag{A.59}$$

Since the slope and the R^2 of this regression are given by

$$b\left(h\right) = \frac{Cov\left(\sum_{j=1}^{h} U_{R,t+j}^2, D_t/P_t\right)}{Var\left[\sum_{j=1}^{h} U_{R,t+j}^2\right]} \quad \text{and} \quad R^2\left(h\right) = \frac{\left(Cov\left(\sum_{j=1}^{h} U_{R,t+j}^2, D_t/P_t\right)\right)^2}{Var\left[\sum_{j=1}^{h} U_{R,t+j}^2\right] Var\left[D_t/P_t\right]} \tag{A.60}$$

and given that

$$Cov\left(\sum_{j=1}^{h} U_{R,t+j}^2, \frac{D_t}{P_t}\right) = \sum_{j=1}^{h} Cov\left(U_{R,t+j}^2, \frac{D_t}{P_t}\right) \quad \text{and} \tag{A.61}$$

$$Var\left[\sum_{j=1}^{h} U_{R,t+j}^2\right] = hVar\left[U_{R,t+1}^2\right] + 2\sum_{j=1}^{h-1}\left(h-j\right) Cov\left(U_{R,t+1}^2, U_{R,t+1+j}^2\right), \tag{A.62}$$

then, to be able to compute analytically the slope and the R^2 of this predictive regression, we need to derive closed-form expressions for $E\left[R_{t+1}\right]$, $Var\left[R_{t+1}\right]$, $Cov\left(R_{t+1}, R_t\right)$, $Var\left[U_{R,t+1}^2\right]$, $Cov\left(U_{R,t+j}^2, D_t/P_t\right)$ and $Cov\left(U_{R,t+1}^2, U_{R,t+1+j}^2\right)$ for $j \geq 1$.

To compute these expressions, observe that:

$$
\begin{aligned}
&Var\left[U_{R,t+1}^2\right] \\
&\quad = \left(1+b_R^4\right) Var\left[R_{t+1}^2\right] + 4b_R^2 Var\left[R_{t+1}R_t\right] + \left(4a_R^2+4a_R^2b_R^2\right) Var\left[R_{t+1}\right] \\
&\qquad + 2b_R^2 Cov\left(R_{t+1}^2, R_t^2\right) - 4b_R Cov\left(R_{t+1}^2, R_{t+1}R_t\right) + 4a_R b_R Cov\left(R_{t+1}^2, R_t\right) \\
&\qquad - 4b_R^3 Cov\left(R_t^2, R_{t+1}R_t\right) - 4a_R b_R^2 Cov\left(R_t^2, R_{t+1}\right) + \left(4a_R b_R^3 - 4a_R\right) Cov\left(R_{t+1}^2, R_{t+1}\right) \\
&\qquad + 8a_R b_R Cov\left(R_{t+1}R_t, R_{t+1}\right) - 8a_R b_R^2 Cov\left(R_{t+1}R_t, R_t\right) - 8a_R^2 b_R Cov\left(R_{t+1}, R_t\right)
\end{aligned}
$$

$$
\begin{aligned}
&Cov\left(U_{R,t+1}^2, U_{R,t+1+j}^2\right) \\
&\quad = \left(1+b_R^4\right) Cov\left(R_{t+1}^2, R_{t+1+j}^2\right) + 4b_R^2 Cov\left(R_{t+1}R_t, R_{t+1+j}R_{t+j}\right) \\
&\qquad + \left(4a_R^2+4a_R^2b_R^2\right) Cov\left(R_{t+1}, R_{t+1+j}\right) + b_R^2\left[Cov\left(R_{t+1}^2, R_{t+j}^2\right) + Cov\left(R_t^2, R_{t+1+j}^2\right)\right] \\
&\qquad - 2b_R\left[Cov\left(R_{t+1}^2, R_{t+1+j}R_{t+j}\right) + Cov\left(R_{t+1}R_t, R_{t+1+j}^2\right)\right] \\
&\qquad + 2a_R b_R\left[Cov\left(R_{t+1}^2, R_{t+j}\right) + Cov\left(R_t, R_{t+1+j}^2\right)\right] \\
&\qquad - 2b_R^3\left[Cov\left(R_t^2, R_{t+1+j}R_{t+j}\right) + Cov\left(R_{t+1}R_t, R_{t+j}^2\right)\right] \\
&\qquad - 2a_R b_R^2\left[Cov\left(R_t^2, R_{t+1+j}\right) + Cov\left(R_{t+1}, R_{t+j}^2\right)\right] \\
&\qquad + \left(2a_R b_R^3 - 2a_R\right)\left[Cov\left(R_t^2, R_{t+j}\right) + Cov\left(R_t, R_{t+j}^2\right)\right] \\
&\qquad + 4a_R b_R\left[Cov\left(R_{t+1}R_t, R_{t+1+j}\right) + Cov\left(R_{t+1}, R_{t+1+j}R_{t+j}\right)\right] \\
&\qquad - 4a_R b_R^2\left[Cov\left(R_{t+1}R_t, R_{t+j}\right) + Cov\left(R_t, R_{t+1+j}R_{t+j}\right)\right] \\
&\qquad - 4a_R^2 b_R\left[Cov\left(R_{t+1}, R_{t+j}\right) + Cov\left(R_t, R_{t+1+j}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
Cov\left(U_{R,t+j}^2, \frac{D_t}{P_t}\right) = {} & Cov\left(R_{t+j}^2, \frac{D_t}{P_t}\right) + b_R^2 Cov\left(R_{t+j-1}^2, \frac{D_t}{P_t}\right) - 2b_R Cov\left(R_{t+j}R_{t+j-1}, \frac{D_t}{P_t}\right) \\
& - 2a_R Cov\left(R_{t+j}, \frac{D_t}{P_t}\right) + 2a_R b_R Cov\left(R_{t+j-1}, \frac{D_t}{P_t}\right)
\end{aligned}
$$

We are able to get all the terms in $Var\left[U_{R,t+1}^2\right]$ and in $Cov\left(U_{R,t+1}^2, U_{R,t+1+j}^2\right),\ j \geq 1$ if we can compute $Cov\left(R_{t+1}^n R_t^m, R_{t+1+j}^q R_{t+j}^p\right),\ j \geq 1$ for given nonnegative integers n, m, q and p. We have:

$$
Cov\left(R_{t+1}^n R_t^m, R_{t+1+j}^q R_{t+j}^p\right) = E\left[R_t^m R_{t+1}^n R_{t+j}^p R_{t+1+j}^q\right] - E\left[R_t^m R_{t+1}^n\right] E\left[R_{t+j}^p R_{t+1+j}^q\right].
$$

Also observe that we can compute $Cov\left(R_{t+1}^n R_t^m, R_{t+1+j}^q R_{t+j}^p\right),\ j \geq 1$ for given nonnegative integers n, m, q and p if we can compute $E\left[R_t^m R_{t+1}^n R_{t+j}^p R_{t+1+j}^q\right],\ j > 1$ for given nonnegative integers n, m, q and p. We have

$$\begin{aligned}
&E\left[R_t^m R_{t+1}^n R_{t+j}^p R_{t+1+j}^q\right] \\
&= E\left[E\left[R_t^m R_{t+1}^n R_{t+j}^p R_{t+1+j}^q \mid \zeta_\tau, \tau \in \mathbb{Z}\right]\right] \\
&= E\left[E\left[R_t^m \mid \zeta_\tau, \tau \in \mathbb{Z}\right] E\left[R_{t+1}^n \mid \zeta_\tau, \tau \in \mathbb{Z}\right] E\left[R_{t+j}^p \mid \zeta_\tau, \tau \in \mathbb{Z}\right] E\left[R_{t+j+1}^q \mid \zeta_\tau, \tau \in \mathbb{Z}\right]\right] \\
&= E\left[\left(\theta_{0m}^\top \zeta_{t-1}\right)\left(\theta_{mn}^\top \zeta_t\right)\left(\theta_{n0}^\top \zeta_{t+1}\right)\left(\theta_{0p}^\top \zeta_{t+j-1}\right)\left(\theta_{pq}^\top \zeta_{t+j}\right)\left(\theta_{q0}^\top \zeta_{t+j+1}\right)\right] \\
&= \theta_{0m}^\top E\left[\zeta_t \zeta_t^\top\right] P^\top \left(\theta_{mn} \odot \left(P^\top \left(\theta_{n0} \odot \left(\left(P^{j-2}\right)^\top \left(\theta_{0p} \odot \left(P^\top \left(\theta_{pq} \odot \left(P^\top \theta_{q0}\right)\right)\right)\right)\right)\right)\right)\right)
\end{aligned}$$

where the second equality comes from the fact that returns are independent conditionally to the Markov chain, and where

$$\theta_{kl} = \lambda_{3d}^k \odot \lambda_{2d}^l \odot \exp\left(l\mu_d + \frac{l^2 \omega_d}{2}\right). \tag{A.63}$$

We can also get all the terms in

$$Cov\left(U_{R,t+j}^2, D_t/P_t\right), \ j \geq 1$$

if we can compute

$$Cov\left(R_{t+j}^n R_{t+j-1}^m, D_t/P_t\right) \quad \text{for} \quad j \geq 1, \quad n \geq 0 \quad \text{and} \quad m \geq 0.$$

We have

$$\begin{aligned}
Cov\left(R_{t+j}^n R_{t+j-1}^m, \frac{D_t}{P_t}\right) &= Cov\left(E\left[R_{t+j}^n R_{t+j-1}^m \mid \zeta_\tau, \tau \in \mathbb{Z}\right], \lambda_{2d}^\top \zeta_t\right) \\
&= Cov\left(E\left[R_{t+j-1}^m \mid \zeta_\tau, \tau \in \mathbb{Z}\right] E\left[R_{t+j}^n \mid \zeta_\tau, \tau \in \mathbb{Z}\right], \lambda_{2d}^\top \zeta_t\right) \\
&= Cov\left(\left(\theta_{0m}^\top \zeta_{t+j-2}\right)\left(\theta_{mn}^\top \zeta_{t+j-1}\right)\left(\theta_{n0}^\top \zeta_{t+j}\right), \lambda_{2d}^\top \zeta_t\right) \\
&= \begin{cases}
\theta_{0m}^\top E\left[\zeta_t \zeta_t^\top\right] P^\top \left(\theta_{mn} \odot \lambda_{2d} \odot \left(P^\top \theta_{n0}\right)\right) \\
\qquad - \left(\theta_{0m}^\top E\left[\zeta_t \zeta_t^\top\right] P^\top \left(\theta_{mn} \odot \left(P^\top \theta_{n0}\right)\right)\right)\left(\lambda_{2d}^\top E\left[\zeta_t\right]\right) & \text{if } j = 1 \\
\\
\lambda_{2d}^\top Var\left[\zeta_t\right] \left(P^{j-2}\right)^\top \left(\theta_{0m} \odot \left(P^\top \left(\theta_{mn} \odot \left(P^\top \theta_{n0}\right)\right)\right)\right) & \text{if } j > 1.
\end{cases}
\end{aligned}$$

Similar formulas can be obtained for consumption growth and dividend growth volatilities using the following table of parameter substitution:

Returns	Consumption Growth	Dividend Growth
λ_{3d}	e	e
λ_{2d}	e	e
μ_d	μ_c	μ_d
ω_d	ω_c	ω_d

D GDA Vs. KP Certainty Equivalents

We have proven that in the long-run risk and recursive utility framework, the GDA certainty equivalent solves both asset pricing puzzles, high predictability of long-horizon excess returns by the dividend-price ratio and the low (or say no) predictability of long-horizon growth rates by the dividend-price ratio, whereas the KP certainty equivalent only solves asset pricing puzzles and produces opposite results in predictability regressions (low or no predictability of returns and high predictability of growth rates by the dividend-price ratio). One might ask how risk-averse is the GDA representative investor compared to the KP one? To answer this question, we compare the indifference curves of the GDA certainty equivalent with $(\gamma = 2.5, \alpha = 0.33, \kappa = 0.985)$ to that of the KP certainty equivalent with $\gamma = 10$ considered by Bansal and Yaron (2004) and Bansal, Kiku and Yaron (2006).

Let Z be an atemporal lottery that put the probability p on the outcome x and $1-p$ on the outcome y. Such a lottery is then characterized by a three-dimensional vector $(x, y, p)^\top$ where $p = Prob\,(x)$. For a given number μ, let focus our attention on all the atemporal lotteries Z such that $\mathcal{R}(Z) = \mu$, that is the indifference set indexed by μ. This set is a surface $S(x, y, p) = 0$ in the space (x, y, p), which for a given y^0 leads to an indifference curve $p = f(x, y^0)$ in the plane (x, p), and for a given p^0 leads to an indifference curve $y = g(x, p^0)$ in the plane (x, y).

With GDA preferences, the indifference set indexed by μ in the space (x, y, p) is the surface characterized by the implicit equation:[3]

$$I_{\alpha,\kappa}\left(\frac{y}{\kappa\mu}\right)\mu^{1-\gamma} - I_{\alpha,1}\left(\frac{y}{\kappa\mu}\right)y^{1-\gamma} - p\left\{\left[I_{\alpha,1}\left(\frac{x}{\kappa\mu}\right)x^{1-\gamma} - I_{\alpha,1}\left(\frac{y}{\kappa\mu}\right)y^{1-\gamma}\right]\right.$$
$$\left. - \left[I_{\alpha,\kappa}\left(\frac{x}{\kappa\mu}\right) - I_{\alpha,\kappa}\left(\frac{y}{\kappa\mu}\right)\right]\mu^{1-\gamma}\right\} = 0.$$

Panels (a) and (b) Figure 1 shows the well-known result that, the more risk-averse is an investor, the more pronounced is the curvature of the indifference curve. In Panel (a), the indifference curve in the plane (x, p) of our GDA investor with $(\gamma = 2.5, \alpha = 0.33, \kappa = 0.985)$ lies in between the indifference curves of KP investors with risk aversions $\gamma = 3$ and $\gamma = 5$ that are less curved than the indifference curve of a KP investor with $\gamma = 10$. Panel (b), shows that the indifference curve in the plane (x, y) of our GDA investor is less curved in the tails compared to that of the KP investor with $\gamma = 10$ and both almost have the same curvature elsewhere. Based on that observation, we argue that our chosen preference parameters for the GDA investor are reasonable if one admits that $\gamma = 10$ is a reasonable upper bound for the risk aversion parameter for KP preferences (Mehra and Prescott (1985)).

[3] The probability p of the outcome x is then given by:

$$p = \frac{I_{\alpha,\kappa}\left(\frac{y}{\kappa\mu}\right)\mu^{1-\gamma} - I_{\alpha,1}\left(\frac{y}{\kappa\mu}\right)y^{1-\gamma}}{\left[I_{\alpha,1}\left(\frac{x}{\kappa\mu}\right)x^{1-\gamma} - I_{\alpha,1}\left(\frac{y}{\kappa\mu}\right)y^{1-\gamma}\right] - \left[I_{\alpha,\kappa}\left(\frac{x}{\kappa\mu}\right) - I_{\alpha,\kappa}\left(\frac{y}{\kappa\mu}\right)\right]\mu^{1-\gamma}}$$

and this is the explicit equation of an indifference curve in the plane (x, y) for a given y.

Figure 1: **Indifference Curves for GDA Preferences**
Indifference curves over two outcomes x and y with the fixed probability $p = Prob(x) = 1/2$.

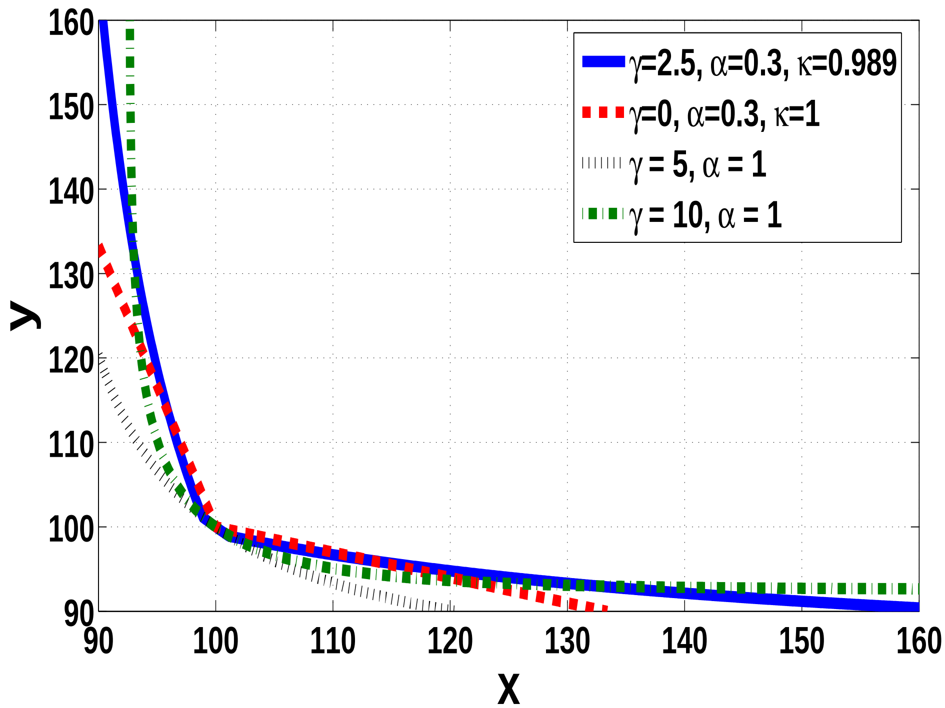

Table 1: **Small Sample Fit of the Long-Run Risk Markov-Switching Model.** In the table, we report and compare moments of simulated annualized consumption and dividend growth rates. Data are simulated from the original LRR model as well as from its Markov-Switching match. Reported statistics are based on $10,000$ simulated samples with 78×12 monthly observations that match the length of the actual data. The entries represent mean, median, 5th, 10th, 90th and 95th percentiles of the monte-carlo distributions of the corresponding statistics.

		mean	5%	10%	50%	90%	95%
$E[\Delta c]$	LRR	1.80	0.82	1.07	1.80	2.55	2.79
	MS	1.80	0.88	1.13	1.86	2.36	2.48
$\sigma[\Delta c]$	LRR	3.25	1.83	2.07	3.18	4.51	4.87
	MS	2.63	1.51	1.62	2.24	4.50	4.82
$AR1(\Delta c)$	LRR	0.16	-0.06	-0.01	0.16	0.33	0.38
	MS	0.22	-0.12	-0.06	0.21	0.51	0.57
$E[\Delta d]$	LRR	1.77	-2.34	-1.32	1.74	4.89	5.86
	MS	1.77	-1.76	-0.80	1.88	4.13	5.01
$\sigma[\Delta d]$	LRR	18.94	10.83	12.09	18.53	26.26	28.29
	MS	14.91	9.15	9.52	11.08	28.14	30.19
$AR1(\Delta d)$	LRR	0.02	-0.18	-0.14	0.02	0.18	0.23
	MS	0.04	-0.17	-0.13	0.04	0.22	0.27
$Corr(\Delta c, \Delta d)$	LRR	0.44	0.26	0.30	0.44	0.57	0.60
	MS	0.46	0.28	0.32	0.47	0.60	0.64