Appendix on the "The Long and the Short of the Risk-Return Trade-Off" *

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Appendix

A Stochastic Discount Factor and Valuation Ratios

Appendix A provides the analytical formulae of financial variables implied by the model at the high frequency level (here daily level). They are proved in Bonomo et al. (2011). The Markov chain is stationary with ergodic distribution and second moments given by:

$$
\begin{aligned}
&E\left[\zeta_t\right] = \mu^\zeta \in \mathbb{R}_+^N, \\
&E\left[\zeta_t \zeta_t^\top\right] = Diag\left(\mu_1^\zeta, .., \mu_N^\zeta\right) \text{ and } \Sigma^\zeta = Var\left[\zeta_t\right] = Diag\left(\mu_1^\zeta, .., \mu_N^\zeta\right) - \mu^\zeta \left(\mu^\zeta\right)^\top,
\end{aligned}
\tag{A.1}
$$

where $Diag\left(u_1, .., u_N\right)$ is the $N \times N$ diagonal matrix whose diagonal elements are u_1,..,u_N.

In order to compute expectations conditional to the Markov chain, we make use of the following results: Let X and Y be two normally distributed random variables with means μ_X and μ_Y, variances σ_X^2 and σ_Y^2, and covariance σ_{XY}. Then, we have

$$
\begin{aligned}
&E\left[\exp\left(uX + vY\right) I\left(X < x\right)\right] \\
&= \exp\left(u\mu_X + v\mu_Y + \frac{1}{2}\left(u^2\sigma_X^2 + 2uv\sigma_{XY} + v^2\sigma_Y^2\right)\right) \Phi\left(\frac{x - \mu_X}{\sigma_X} - u\sigma_X - v\frac{\sigma_{XY}}{\sigma_X}\right),
\end{aligned}
\tag{A.2}
$$

where $\Phi\left(\cdot\right)$ is the standard normal cumulative distribution function.

We then show that the stochastic discount factor $M_{t,t+\Delta}$ can also be written as

$$
M_{t,t+\Delta} = \delta_{t,t+\Delta}^* \exp\left(-\gamma g_{c,t+\Delta}\right)\left[1 + \ell I\left(g_{c,t+\Delta} < -g_{v,t+\Delta} + \ln\theta\right)\right] \tag{A.3}
$$

where

$$
\ln \delta_{t,t+\Delta}^* = \zeta_t^\top A \zeta_{t+\Delta} \quad \text{and} \quad g_{v,t+\Delta} = \zeta_t^\top B \zeta_{t+\Delta} \tag{A.4}
$$

and where the components of matrices A and B are explicitly defined by

$$
\begin{aligned}
a_{ij} &= \ln\delta + \left(\frac{1}{\psi} - \gamma\right) b_{ij} - \ln\left[1 + \ell\theta^{1-\gamma}\sum_{j=1}^{N} p_{ij}\Phi\left(q_{ij}\right)\right] \\
b_{ij} &= \ln\left(\frac{\lambda_{1v,j}}{\lambda_{1z,i}}\right) \quad \text{and} \quad q_{ij} = \frac{-b_{ij} + \ln\theta - \mu_{c,i}}{\sqrt{\omega_{c,i}}}.
\end{aligned}
\tag{A.5}
$$

Likewise, one has:

Proposition A.1 ***Characterization of Welfare Valuation Ratios.*** *Let*

$$\frac{\mathcal{R}_t\left(V_{t+\Delta}\right)}{C_t} = \lambda_{1z}^{\top}\zeta_t \text{ and } \frac{V_t}{C_t} = \lambda_{1v}^{\top}\zeta_t$$

respectively denote the ratio of the certainty equivalent of future lifetime utility to current consumption and the ratio of lifetime utility to consumption. The components of the vectors λ_{1z} and λ_{1v} are given by:

$$\lambda_{1z,i} = \exp\left(\mu_{c,i} + \frac{1-\gamma}{2}\omega_{c,i}\right)\left(\sum_{j=1}^{N} p_{ij}^{*}\lambda_{1v,j}^{1-\gamma}\right)^{\frac{1}{1-\gamma}} \tag{A.6}$$

$$\lambda_{1v,i} = \left\{(1-\delta) + \delta\lambda_{1z,i}^{1-\frac{1}{\psi}}\right\}^{\frac{1}{1-\frac{1}{\psi}}} \text{ if } \psi \neq 1 \text{ and } \lambda_{1v,i} = \lambda_{1z,i}^{\delta} \text{ if } \psi = 1, \tag{A.7}$$

where the matrix $P^{*\top} = \left[p_{ij}^{*}\right]_{1\leq i,j\leq N}$ is defined in (A.11).

In the following, $\odot$ denotes the component-by-component multiplication operator.

Proposition A.2 ***Characterization of Asset Prices.*** *Let*

$$\frac{P_{d,t}}{D_t} = \lambda_{1d}^{\top}\zeta_t, \quad \frac{P_{c,t}}{C_t} = \lambda_{1c}^{\top}\zeta_t \quad \text{and} \quad R_{f,t+\Delta} = \frac{1}{\lambda_{1f}^{\top}\zeta_t}$$

respectively denote the price-dividend ratio, the price-consumption ratio and the risk-free rate. The components of the vectors λ_{1d}, λ_{1c}, and λ_{1f} are given by:

$$\lambda_{1d,i} = \delta\left(\frac{1}{\lambda_{1z,i}}\right)^{\frac{1}{\psi}-\gamma}\exp\left(\mu_{cd,i} + \frac{\omega_{cd,i}}{2}\right)\left(\lambda_{1v}^{\frac{1}{\psi}-\gamma}\right)^{\top} P^{**}\left(Id - \delta A^{**}\left(\mu_{cd} + \frac{\omega_{cd}}{2}\right)\right)^{-1} e_i \tag{A.8}$$

$$\lambda_{1c,i} = \delta\left(\frac{1}{\lambda_{1z,i}}\right)^{\frac{1}{\psi}-\gamma}\exp\left(\mu_{cc,i} + \frac{\omega_{cc,i}}{2}\right)\left(\lambda_{1v}^{\frac{1}{\psi}-\gamma}\right)^{\top} P^{*}\left(Id - \delta A^{*}\left(\mu_{cc} + \frac{\omega_{cc}}{2}\right)\right)^{-1} e_i \tag{A.9}$$

$$\lambda_{1f,i} = \delta\exp\left(-\gamma\mu_{c,i} + \frac{\gamma^2}{2}\omega_{c,i}\right)\sum_{j=1}^{N}\tilde{p}_{ij}^{*}\left(\frac{\lambda_{1v,j}}{\lambda_{1z,i}}\right)^{\frac{1}{\psi}-\gamma} \tag{A.10}$$

where the vectors $\mu_{cd} = -\gamma\mu_c + \mu_d$, $\omega_{cd} = \omega_c + \omega_d - 2\gamma\rho \odot \sqrt{\omega_c} \odot \sqrt{\omega_d}$, $\mu_{cc} = (1-\gamma)\mu_c$, $\omega_{cc} = (1-\gamma)^2\omega_c$, and the matrices $P^{**\top} = \left[p_{ij}^{**}\right]_{1\leq i,j\leq N}$ and $\tilde{P}^{*\top} = \left[\tilde{p}_{ij}^{*}\right]_{1\leq i,j\leq N}$ as well as the matrix functions $A^{**}(u)$ and $A^{*}(u)$ are defined in (A.13), (A.14), (A.12) and (A.15), respectively. The *vector e_i denotes the $N \times 1$ vector with all components equal to zero but the ith component is equal to one.*

The components of the matrix $P^{*\top} = \left[p^*_{ij}\right]_{1 \leq i,j \leq N}$ in (A.6) and (A.9), and the matrix function $A^*(u)$ also in (A.9) are defined by:

$$p^*_{ij} = p_{ij} \frac{1 + \ell\Phi\left(q_{ij} - (1-\gamma)\sqrt{\omega_{c,i}}\right)}{1 + \ell\theta^{1-\gamma} \sum_{j=1}^{N} p_{ij}\Phi\left(q_{ij}\right)} \tag{A.11}$$

$$A^*(u) = Diag\left(\exp\left(\left(\frac{1}{\psi} - \gamma\right) b_{11} + u_1\right), ..., \exp\left(\left(\frac{1}{\psi} - \gamma\right) b_{NN} + u_N\right)\right) P^*, \tag{A.12}$$

where $\Phi(\cdot)$ denotes the cumulative distribution function of a standard normal random variable.

The matrix $P^{**\top} = \left[p^{**}_{ij}\right]_{1 \leq i,j \leq N}$ in (A.8), and the matrix $\tilde{P}^{*\top} = \left[\tilde{p}^*_{ij}\right]_{1 \leq i,j \leq N}$ in (A.10) have their components given by:

$$p^{**}_{ij} = p_{ij} \frac{1 + \ell\Phi\left(q_{ij} - \left(\rho_i\sqrt{\omega_{d,i}} - \gamma\sqrt{\omega_{c,i}}\right)\right)}{1 + \ell\theta^{1-\gamma} \sum_{j=1}^{N} p_{ij}\Phi\left(q_{ij}\right)} \tag{A.13}$$

$$\tilde{p}^*_{ij} = p_{ij} \frac{1 + \ell\Phi\left(q_{ij} + \gamma\sqrt{\omega_{c,i}}\right)}{1 + \ell\theta^{1-\gamma} \sum_{j=1}^{N} p_{ij}\Phi\left(q_{ij}\right)}. \tag{A.14}$$

The matrix function $A^{**}(u)$ in (A.8) is defined by:

$$A^{**}(u) = Diag\left(\exp\left(\left(\frac{1}{\psi} - \gamma\right) b_{11} + u_1\right), ..., \exp\left(\left(\frac{1}{\psi} - \gamma\right) b_{NN} + u_N\right)\right) P^{**}. \tag{A.15}$$

B Time Aggregation: Model-Implied Low Frequency Moments

Appendix B provides the first and second moments of the vector $\left(g_{c,t,t+h},\ g_{d,t,t+h},\ z_{d,t,t+h},\ r_{f,t,t+h},\ r_{t,t+h}\right)^\top$, that is, consumption growth, dividend growth, log price-dividend ratio, log risk-free return and the excess log equity-return at lower frequencies, like monthly, which are implied by the model defined at the high frequency (here daily).

Given the postulated dynamics of endowment and the implied-dynamics of asset prices at the frequency Δ (at which several economic variables may be unobservable), we are interested in the properties of these quantities at lower frequencies. We defined frequency h consumption growth, dividend growth, log price-dividend ratio, log risk-free return and excess log equity return over the log risk free return as follows:

$$\begin{aligned}
g_{c,t,t+h} &= \ln\left(\frac{C_{t,t+h}}{C_{t-h,t}}\right), \quad g_{d,t,t+h} = \ln\left(\frac{D_{t,t+h}}{D_{t-h,t}}\right) \quad \text{and} \quad z_{d,t,t+h} = \ln\left(\frac{\bar{P}_{d,t,t+h}}{D_{t,t+h}}\right) \\
r_{f,t,t+h} &= \sum_{j=1}^{h/\Delta} r_{f,t+j\Delta} \quad \text{and} \quad r_{t,t+h} = \sum_{j=1}^{h/\Delta} r_{t+j\Delta}
\end{aligned} \tag{B.1}$$

where $r_{f,t+\Delta} = \nu_{1f}^{\top}\zeta_t$ with $\nu_{1f} = \ln \lambda_{1f}$, and where

$$C_{t,t+h} = \sum_{i=1}^{h/\Delta} C_{t+i\Delta}, \quad D_{t,t+h} = \sum_{i=1}^{h/\Delta} D_{t+i\Delta} \quad \text{and} \quad \bar{P}_{d,t,t+h} = \frac{1}{h/\Delta}\sum_{i=1}^{h/\Delta} P_{d,t+i\Delta}. \tag{B.2}$$

We show that

$$\begin{aligned}
g_{c,t,t+h} &\approx g_{c,t+\Delta} + \sum_{j=1}^{h/\Delta-1}\left(1-\frac{j}{h/\Delta}\right)\left(g_{c,t+\Delta+j\Delta} + g_{c,t+\Delta-j\Delta}\right) \\
g_{d,t,t+h} &\approx g_{d,t+\Delta} + \sum_{j=1}^{h/\Delta-1}\left(1-\frac{j}{h/\Delta}\right)\left(g_{d,t+\Delta+j\Delta} + g_{d,t+\Delta-j\Delta}\right) \\
z_{d,t,t+h} &\approx -\ln\left(h/\Delta\right) + \frac{1}{h/\Delta}\sum_{j=1}^{h/\Delta} z_{d,t+j\Delta} \quad \text{where} \quad z_{d,t} = \ln\left(\frac{P_{d,t}}{D_t}\right) \\
&= \sum_{j=1}^{h/\Delta} z_{d,h,t+j\Delta} \quad \text{where} \quad z_{d,h,t} = \frac{z_{d,t} - \ln\left(h/\Delta\right)}{h/\Delta}.
\end{aligned} \tag{B.3}$$

It follows that first and second moments of the low frequency vector process

$$L_{t,t+h} = \begin{pmatrix} g_{c,t,t+h} & g_{d,t,t+h} & z_{d,t,t+h} & r_{f,t,t+h} & r_{t,t+h} \end{pmatrix}^{\top}$$

are completely determined by those of the high frequency vector process

$$H_t = \begin{pmatrix} g_{c,t} & g_{d,t} & z_{d,h,t} & r_{f,t} & r_t \end{pmatrix}^{\top}.$$

The mean and the autocovariance matrices of the vector process H_t are defined by $\mu^H = E\left[H_t\right] = (\mu_1^H,\ \mu_2^H,\ \mu_3^H,\ \mu_4^H,\ \mu_5^H)^{\top}$ and

$$\Gamma^{H}(j) = Cov\left(H_t, H_{t+j\Delta}\right) = \begin{bmatrix} \gamma_{11}^{H}(j) & \gamma_{12}^{H}(j) & \gamma_{13}^{H}(j) & \gamma_{14}^{H}(j) & \gamma_{15}^{H}(j) \\ \gamma_{21}^{H}(j) & \gamma_{22}^{H}(j) & \gamma_{23}^{H}(j) & \gamma_{24}^{H}(j) & \gamma_{25}^{H}(j) \\ \gamma_{31}^{H}(j) & \gamma_{32}^{H}(j) & \gamma_{33}^{H}(j) & \gamma_{34}^{H}(j) & \gamma_{35}^{H}(j) \\ \gamma_{41}^{H}(j) & \gamma_{42}^{H}(j) & \gamma_{43}^{H}(j) & \gamma_{44}^{H}(j) & \gamma_{45}^{H}(j) \\ \gamma_{51}^{H}(j) & \gamma_{52}^{H}(j) & \gamma_{53}^{H}(j) & \gamma_{54}^{H}(j) & \gamma_{55}^{H}(j) \end{bmatrix}. \tag{B.4}$$

We have

$$\begin{aligned} &\mu_1^H = \mu_c^\top \mu^\zeta, \quad \mu_2^H = \mu_d^\top \mu^\zeta \quad \text{and} \quad \mu_3^H = \nu_{1d,h}^\top \mu^\zeta, \quad \text{where} \quad \nu_{1d,h} = \frac{\nu_{1d} - \ln(h/\Delta)}{h/\Delta}, \\ &\mu_4^H = \nu_{1f}^\top \mu^\zeta \quad \text{and} \quad \mu_5^H = \mu_r^\top \mu^\zeta, \quad \text{with} \quad \nu_{1d} = \ln \lambda_{1d}, \end{aligned} \tag{B.5}$$

where μ_r is the diagonal of ΛP, and $\forall j \geq 0$ we have

$$\begin{aligned}
&\gamma_{11}^{H}(j) = \mu_c^\top P^j \Sigma^\zeta \mu_c + \left(\omega_c^\top \mu^\zeta\right) I(j=0) \\
&\gamma_{12}^{H}(j) = \mu_d^\top P^j \Sigma^\zeta \mu_c + \left(\left(\rho \odot \sqrt{\omega_c} \odot \sqrt{\omega_d}\right)^\top \mu^\zeta\right) I(j=0) \\
&\gamma_{13}^{H}(j) = \nu_{1d,h}^\top P^{j+1} \Sigma^\zeta \mu_c \quad \text{and} \quad \gamma_{14}^{H}(j) = \nu_{1f}^\top P^j \Sigma^\zeta \mu_c \\
&\gamma_{15}^{H}(j) = \mu_r^\top P^j \Sigma^\zeta \mu_c + \left(\left(\rho \odot \sqrt{\omega_c} \odot \sqrt{\omega_d}\right)^\top \mu^\zeta\right) I(j=0) \\
&\gamma_{21}^{H}(j) = \mu_c^\top P^j \Sigma^\zeta \mu_d + \left(\left(\rho \odot \sqrt{\omega_c} \odot \sqrt{\omega_d}\right)^\top \mu^\zeta\right) I(j=0) \\
&\gamma_{22}^{H}(j) = \mu_d^\top P^j \Sigma^\zeta \mu_d + \left(\omega_d^\top \mu^\zeta\right) I(j=0) \\
&\gamma_{23}^{H}(j) = \nu_{1d,h}^\top P^{j+1} \Sigma^\zeta \mu_d \quad \text{and} \quad \gamma_{24}^{H}(j) = \nu_{1f}^\top P^j \Sigma^\zeta \mu_d \\
&\gamma_{25}^{H}(j) = \mu_r^\top P^j \Sigma^\zeta \mu_d + \left(\left(\rho \odot \sqrt{\omega_c} \odot \sqrt{\omega_d}\right)^\top \mu^\zeta\right) I(j=0) \\
&\gamma_{31}^{H}(j) = \left(\nu_{1d,h}^\top P \Sigma^\zeta \mu_c\right) I(j=0) + \left(\mu_c^\top P^{j-1} \Sigma^\zeta \nu_{1d,h}\right) I(j \geq 1) \\
&\gamma_{32}^{H}(j) = \left(\nu_{1d,h}^\top P \Sigma^\zeta \mu_d\right) I(j=0) + \left(\mu_d^\top P^{j-1} \Sigma^\zeta \nu_{1d,h}\right) I(j \geq 1) \\
&\gamma_{33}^{H}(j) = \nu_{1d,h}^\top P^j \Sigma^\zeta \nu_{1d,h} \quad \text{and} \quad \gamma_{34}^{H}(j) = \left(\nu_{1d,h}^\top P \Sigma^\zeta \nu_{1f}\right) I(j=0) + \left(\nu_{1f}^\top P^{j-1} \Sigma^\zeta \nu_{1d,h}\right) I(j \geq 1) \\
&\gamma_{35}^{H}(j) = \left(\breve{\nu}_{1d,h,1}^\top \mu^\zeta - \left(\nu_{1d,h}^\top \mu^\zeta\right)\left(\mu_r^\top \mu^\zeta\right)\right) I(j=0) + \left(\mu_r^\top P^{j-1} \Sigma^\zeta \nu_{1d,h}\right) I(j \geq 1) \\
&\gamma_{41}^{H}(j) = \mu_c^\top P^j \Sigma^\zeta \nu_{1f}, \quad \gamma_{42}^{H}(j) = \mu_d^\top P^j \Sigma^\zeta \nu_{1f} \quad \text{and} \quad \gamma_{43}^{H}(j) = \nu_{1d,h}^\top P^{j+1} \Sigma^\zeta \nu_{1f}
\end{aligned} \tag{B.6}$$

$$\gamma_{44}^{H}(j)=\nu_{1f}^{\top}P^{j}\Sigma^{\zeta}\nu_{1f} \quad \text{and} \quad \gamma_{45}^{H}(j)=\mu_{r}^{\top}P^{j}\Sigma^{\zeta}\nu_{1f}$$

$$\gamma_{51}^{H}(j)=\left(\mu_{c}^{\top}\Sigma^{\zeta}\mu_{r}+\left(\rho\odot\sqrt{\omega_{c}}\odot\sqrt{\omega_{d}}\right)^{\top}\mu^{\zeta}\right)I(j=0)+\left(\breve{\mu}_{c,j}^{\top}\mu^{\zeta}-\left(\mu_{c}^{\top}\mu^{\zeta}\right)\left(\mu_{r}^{\top}\mu^{\zeta}\right)\right)I(j\geq 1)$$

$$\gamma_{52}^{H}(j)=\left(\mu_{d}^{\top}\Sigma^{\zeta}\mu_{r}+\omega_{d}^{\top}\mu^{\zeta}\right)I(j=0)+\left(\breve{\mu}_{d,j}^{\top}\mu^{\zeta}-\left(\mu_{d}^{\top}\mu^{\zeta}\right)\left(\mu_{r}^{\top}\mu^{\zeta}\right)\right)I(j\geq 1)$$

$$\gamma_{53}^{H}(j)=\breve{\nu}_{1d,h,j+1}^{\top}\mu^{\zeta}-\left(\nu_{1d,h}^{\top}\mu^{\zeta}\right)\left(\mu_{r}^{\top}\mu^{\zeta}\right)$$

$$\gamma_{54}^{H}(j)=\left(\nu_{1f}^{\top}\Sigma^{\zeta}\mu_{r}\right)I(j=0)+\left(\breve{\nu}_{1f,j}^{\top}\mu^{\zeta}-\left(\mu_{d}^{\top}\mu^{\zeta}\right)\left(\mu_{r}^{\top}\mu^{\zeta}\right)\right)I(j\geq 1)$$

$$\gamma_{55}^{H}(j)=\left(\left(\mu_{r}^{(2)}+\omega_{d}\right)^{\top}\mu^{\zeta}-\left(\mu_{r}^{\top}\mu^{\zeta}\right)^{2}\right)I(j=0)+\left(\breve{\mu}_{r,j}^{\top}\mu^{\zeta}-\left(\mu_{r}^{\top}\mu^{\zeta}\right)^{2}\right)I(j\geq 1)$$

with $\gamma_{nq}^{H}(-j)=\gamma_{qn}^{H}(j)$ for $n,q\in\{1,2,3,4,5\}$, where of a given vector u, we have $\breve{u}_{j}$ is the diagonal of the matrix $\left(\left(eu^{\top}P^{j-1}\right)\odot\Lambda\right)P$ and where $\mu_{r}^{(2)}$ is the diagonal of the matrix $(\Lambda\odot\Lambda)\,P$.

We show that the components of the mean of the vector process $L_{t,t+h}$ are given by:

$$\mu_{i}^{L}=(h/\Delta)\,\mu_{i}^{H}, \quad \text{for} \quad i\in\{1,2,3,4,5\}. \tag{B.7}$$

The 5×5 autocovariance matrices of the vector process $L_{t,t+h}$ are defined by

$$\Gamma^{L}(k)=Cov\left(L_{t,t+h},L_{t+kh,t+(k+1)h}\right). \tag{B.8}$$

We have:

$$\begin{aligned}
\gamma_{nq}^{L}(k)=&\;\gamma_{nq}^{H}\left(\frac{kh}{\Delta}\right)+2\sum_{j=1}^{h/\Delta-1}\left(1-\frac{j}{h/\Delta}\right)\left(\gamma_{nq}^{H}\left(\frac{kh}{\Delta}+j\right)+\gamma_{nq}^{H}\left(\frac{kh}{\Delta}-j\right)\right)\\
&+\sum_{j=1}^{h/\Delta-1}\sum_{i=1}^{h/\Delta-1}\left(1-\frac{j}{h/\Delta}\right)\left(1-\frac{i}{h/\Delta}\right)\left(\gamma_{nq}^{H}\left(\frac{kh}{\Delta}+i-j\right)+\gamma_{nq}^{H}\left(\frac{kh}{\Delta}-i-j\right)\right.\\
&\qquad\left.\gamma_{nq}^{H}\left(\frac{kh}{\Delta}+i+j\right)+\gamma_{nq}^{H}\left(\frac{kh}{\Delta}-i+j\right)\right)\\
\gamma_{nl}^{L}(k)=&\sum_{i=1}^{h/\Delta}\gamma_{nl}^{H}\left(\frac{kh}{\Delta}+i-1\right)\\
&+\sum_{j=1}^{h/\Delta-1}\sum_{i=1}^{h/\Delta}\left(1-\frac{j}{h/\Delta}\right)\left(\gamma_{nl}^{H}\left(\frac{kh}{\Delta}+i-j-1\right)+\gamma_{nl}^{H}\left(\frac{kh}{\Delta}+i+j-1\right)\right)\\
\gamma_{lq}^{L}(k)=&\sum_{i=1}^{h/\Delta}\gamma_{lq}^{H}\left(\frac{kh}{\Delta}-i+1\right)
\end{aligned} \tag{B.9}$$

$$+\sum_{j=1}^{h/\Delta-1}\sum_{i=1}^{h/\Delta}\left(1-\frac{j}{h/\Delta}\right)\left(\gamma_{lq}^{H}\left(\frac{kh}{\Delta}-i+j+1\right)+\gamma_{lq}^{H}\left(\frac{kh}{\Delta}-i-j+1\right)\right)$$

for all $n,q\in\{1,2\}$ and $l\in\{3,4,5\}$.

We also have

$$\gamma_{nq}^{L}(k)=\frac{h}{\Delta}\gamma_{nq}^{H}\left(\frac{kh}{\Delta}\right)+\sum_{j=1}^{h/\Delta-1}\left(\frac{h}{\Delta}-j\right)\left(\gamma_{nq}^{H}\left(\frac{kh}{\Delta}+j\right)+\gamma_{nq}^{H}\left(\frac{kh}{\Delta}-j\right)\right) \tag{B.10}$$

for all $n,q\in\{3,4,5\}$.

C Risk-Return Trade-off

The autocovariance matrices of the vector process $X_{t-1,t}$ are defined by

$$\Gamma^{X}(l)=Cov\left(X_{t-1,t},X_{t+l-1,t+l}\right)=\begin{bmatrix}\gamma_{11}^{X}(l) & \gamma_{12}^{X}(l)\\ \gamma_{21}^{X}(l) & \gamma_{22}^{X}(l)\end{bmatrix}. \tag{C.1}$$

The variances of long-horizon returns and long-horizon realized variance, as well as their covariances, can be expressed as follows:

$$Var\left[\begin{pmatrix} r_{t,t+h}\\ \sigma_{t,t+h}^{2}\end{pmatrix}\right]=Var\left[\begin{pmatrix} r_{t-h,t}\\ \sigma_{t-h,t}^{2}\end{pmatrix}\right]=h\Gamma^{X}(0)+\sum_{l=1}^{h-1}(h-l)\left(\Gamma^{X}(l)+\Gamma^{X}(l)^{\top}\right). \tag{C.2}$$

The covariance of future long-horizon returns with past long-horizon realized variance can be expressed as follows:

$$Cov\left(\sigma_{t-m,t}^{2},r_{t,t+h}\right)=\min(m,h)\sum_{l=\min(m,h)}^{\max(m,h)}\gamma_{21}^{X}(l)+\sum_{l=1}^{\min(m,h)-1}l\left(\gamma_{21}^{X}(l)+\gamma_{21}^{X}(m+h-l)\right). \tag{C.3}$$

The covariance of past long-horizon returns with future long-horizon realized variance can be expressed as follows:

$$Cov\left(r_{t-m,t},\sigma_{t,t+h}^{2}\right)=\min(m,h)\sum_{l=\min(m,h)}^{\max(m,h)}\gamma_{12}^{X}(l)+\sum_{l=1}^{\min(m,h)-1}l\left(\gamma_{12}^{X}(l)+\gamma_{12}^{X}(m+h-l)\right). \tag{C.4}$$

We also have that $\forall l$ and $\forall n,q\in\{1,2\}$,

$$\gamma_{nq}^{X}(l)=\frac{1}{\Delta}\gamma_{nq}^{Y}\left(\frac{l}{\Delta}\right)+\sum_{j=1}^{1/\Delta-1}\left(\frac{1}{\Delta}-j\right)\left(\gamma_{nq}^{Y}\left(\frac{l}{\Delta}+j\right)+\gamma_{nq}^{Y}\left(\frac{l}{\Delta}-j\right)\right). \tag{C.5}$$

D Leverage and Volatility Feedback Effects

The autocovariance matrices of the vector process Y_t are defined by

$$\Gamma^{Y}(j)=Cov\left(Y_t,Y_{t+j\Delta}\right)=\begin{bmatrix}\gamma_{11}^{Y}(j) & \gamma_{12}^{Y}(j)\\ \gamma_{21}^{Y}(j) & \gamma_{22}^{Y}(j)\end{bmatrix}. \tag{D.1}$$

We recall the property $\forall j\geq 0,\ E_t\left[\zeta_{t+j\Delta}\right]=P^{j}\zeta_t$. Let $Y_t^{(n)}$ denotes the nth component of the vector process Y_t, for example $Y_t^{(2)}\equiv r_t^2$.

We now adopt the following notations, $\forall n,q\in\{1,2\}$:

$$\begin{aligned}E_t\left[Y_{t+\Delta}^{(n)}\mid\zeta_{k\Delta},k\in\mathbb{Z}\right]&=\zeta_t^{\top}U^{(n)}\zeta_{t+\Delta},\\ E_t\left[Y_{t+\Delta}^{(n)}Y_{t+\Delta}^{(q)}\mid\zeta_{k\Delta},k\in\mathbb{Z}\right]&=\zeta_t^{\top}U^{(nq)}\zeta_{t+\Delta}.\end{aligned} \tag{D.2}$$

We show that:

$$U^{(1)}=\Lambda\ \text{ and }\ U^{(2)}=(\Lambda\odot\Lambda)+\omega_d e^{\top}. \tag{D.3}$$

We also show that:

$$\begin{aligned}U^{(11)}&=(\Lambda\odot\Lambda)+\omega_d e^{\top}\\ U^{(12)}&=U^{(21)}=(\Lambda\odot\Lambda\odot\Lambda)+3\Lambda\odot\left(\omega_d e^{\top}\right)\\ U^{(22)}&=(\Lambda\odot\Lambda\odot\Lambda\odot\Lambda)+6\left(\Lambda\odot\Lambda\right)\odot\left(\omega_d e^{\top}\right)+3\left(\omega_d\odot\omega_d\right)e^{\top}.\end{aligned} \tag{D.4}$$

We also adopt the following notations, $\forall n,q\in\{1,2\}$:

$$\begin{aligned}E_t\left[Y_{t+\Delta+j\Delta}^{(n)}\right]&=\left(\Psi_0^{(n)}\right)^{\top}P^{j}\zeta_t,\\ E_t\left[Y_{t+\Delta}^{(n)}Y_{t+\Delta+j\Delta}^{(q)}\right]&=\left(\Psi_j^{(nq)}\right)^{\top}\zeta_t,\quad\forall j\geq 0.\end{aligned} \tag{D.5}$$

We show that, $\forall n, q \in \{1,2\}$:

$$
\begin{aligned}
&\Psi_0^{(n)} \text{ is the diagonal of the matrix } U^{(n)}P, \\
&\Psi_0^{(nq)} \text{ is the diagonal of the matrix } U^{(nq)}P, \\
&\Psi_j^{(nq)} \text{ is the diagonal of the matrix } \left(U^{(n)} \odot \left(e\left(\Psi_0^{(q)}\right)^\top P^{j-1}\right)\right)P, \quad \forall j \geq 1.
\end{aligned}
\tag{D.6}
$$

Finally we have that, $\forall n, q \in \{1,2\}$:

$$
\begin{aligned}
\mu_n^Y &= E\left[Y_t^{(n)}\right] = \left(\Psi_0^{(n)}\right)^\top \mu^\zeta, \\
\gamma_{nq}^Y(j) &= \left(\left(\Psi_j^{(nq)}\right)^\top \mu^\zeta\right) - \left(\left(\Psi_0^{(n)}\right)^\top \mu^\zeta\right)\left(\left(\Psi_0^{(q)}\right)^\top \mu^\zeta\right), \quad \forall j \geq 0.
\end{aligned}
\tag{D.7}
$$

E Variance Premium

E.1 Dynamics under the $\mathbb{Q}$-measure

Henceforth, dynamics under the risk-neutral ($\mathbb{Q}$) measure will be represented with $\mathbb{Q}$ subscript.

Dynamics of the Markov-chain: We have

$$
\begin{aligned}
E_t^{\mathbb{Q}}\left[\zeta_{t+\Delta}\right] &= E_t\left[M_{t,t+\Delta} R_{f,t+\Delta} \zeta_{t+\Delta}\right] \\
&= \dots \\
&= E_t\left[\zeta_{t+\Delta}\zeta_{t+\Delta}^\top\right]\left(\tilde{M} \odot \left(\lambda_{2f} e^\top\right)\right)^\top \zeta_t \\
&= \left(Diag\left(e_1^\top P\zeta_t, \dots, e_N^\top P\zeta_t\right)\right)\left(\tilde{M} \odot \left(\lambda_{2f} e^\top\right)\right)^\top \zeta_t \\
&= \dots \\
&= \mathcal{E}\left(\left(\tilde{M} \odot \left(\lambda_{2f} e^\top\right)\right)^\top \otimes P\right)\mathcal{E}^\top \zeta_t
\end{aligned}
\tag{E.1}
$$

where $\mathcal{E}$ is the $N \times N^2$ matrix such that the ith row is the vector $(e_i \otimes e_i)^\top$, where the components of the matrix $\tilde{M}$ are given by:

$$
\tilde{m}_{ij} = \exp\left(a_{ij} - \gamma\mu_{c,i} + \frac{1}{2}\gamma^2\omega_{c,i}\right)\left[1 + \ell\Phi\left(q_{ij} + \gamma\sqrt{\omega_{c,i}}\right)\right], \tag{E.2}
$$

and where $\lambda_{2f} = 1/\lambda_{1f}$. It follows that, under the risk-neutral measure, the Markov chain s_t has the one-period transition probability matrix

$$P^{\mathbb{Q}} = \mathcal{E}\left(\left(\tilde{M} \odot \left(\lambda_{2f} e^{\top}\right)\right)^{\top} \otimes P\right)\mathcal{E}^{\top}.$$

Let $P^{\mathbb{Q}(j)}$ be the j-period transition probability matrix under the risk neutral measure, defined by

$$E_t^{\mathbb{Q}}\left[\zeta_{t+j\Delta}\right] = P^{\mathbb{Q}(j)}\zeta_t. \tag{E.3}$$

We show that $P^{\mathbb{Q}(j)},\ j \geq 1$ satisfies the recursion

$$P^{\mathbb{Q}(j)} = P^{\mathbb{Q}(j-1)}\mathcal{E}\left(\left(\lambda_{1f}^{(j-1)}\left(\lambda_{1f}^{(1)} \odot \lambda_{2f}^{(j)}\right)^{\top}\right) \otimes P^{\mathbb{Q}}\right)\mathcal{E}^{\top} \text{ with } P^{\mathbb{Q}(1)} = P^{\mathbb{Q}},$$

where $\lambda_{1f}^{(j)}$ is the vector of j-period risk-free bond prices, defined by

$$E_t\left[M_{t,t+j\Delta}\right] = \lambda_{1f}^{(j)\top}\zeta_t, \tag{E.4}$$

and satisfying the recursion

$$\lambda_{1f}^{(j)} = \lambda_{1f} \odot \left(P^{\mathbb{Q}\top}\lambda_{1f}^{(j-1)}\right) \text{ with } \lambda_{1f}^{(1)} = \lambda_{1f}.$$

Dynamics of the returns and squared returns: We adopt the following notation, $\forall n \in \{1, 2\}$:

$$E_t\left[M_{t,t+\Delta}Y_{t+\Delta}^{(n)} \mid \zeta_{k\Delta}, k \in \mathbb{Z}\right] = \zeta_t^{\top}U^{\mathbb{Q}(n)}\zeta_{t+\Delta}. \tag{E.5}$$

We show that

$$\begin{aligned} U^{\mathbb{Q}(1)} = \exp\left(A - \gamma\mu_c e^{\top} + \frac{\gamma^2}{2}\omega_c e^{\top}\right) \odot \Big[\left(\Lambda - \gamma\left(\rho \odot \sqrt{\omega_c} \odot \sqrt{\omega_d}\right)e^{\top}\right) \odot \left(1 + \ell\Phi\left(Q + \gamma\sqrt{\omega_c}e^{\top}\right)\right) \\ -\ell\left(\left(\rho \odot \sqrt{\omega_d}\right)e^{\top}\right) \odot \phi\left(Q + \gamma\sqrt{\omega_c}e^{\top}\right)\Big] \end{aligned} \tag{E.6}$$

$$
\begin{aligned}
U^{\mathbb{Q}(2)} = \exp\left(A - \gamma\mu_c e^\top + \frac{\gamma^2}{2}\omega_c e^\top\right) \odot \\
&\left[\left(\omega_d e^\top + \left(\Lambda - \gamma\left(\rho \odot \sqrt{\omega_c} \odot \sqrt{\omega_d}\right) e^\top\right) \odot \left(\Lambda - \gamma\left(\rho \odot \sqrt{\omega_c} \odot \sqrt{\omega_d}\right) e^\top\right)\right) \odot \right. \\
&\quad \left(1 + \ell\Phi\left(Q + \gamma\sqrt{\omega_c} e^\top\right)\right) \\
&- 2\ell\left(\left(\rho \odot \sqrt{\omega_d}\right) e^\top\right) \odot \left(\Lambda - \gamma\left(\rho \odot \sqrt{\omega_c} \odot \sqrt{\omega_d}\right) e^\top\right) \odot \phi\left(Q + \gamma\sqrt{\omega_c} e^\top\right) \\
&\left. -\ell\left(\left(\rho \odot \sqrt{\omega_d}\right) e^\top\right) \odot \left(\left(\rho \odot \sqrt{\omega_d}\right) e^\top\right) \odot \left(Q + \gamma\sqrt{\omega_c} e^\top\right) \odot \phi\left(Q + \gamma\sqrt{\omega_c} e^\top\right)\right].
\end{aligned}
\tag{E.7}
$$

We also adopt the following notation, $\forall n \in \{1, 2\}$:

$$
E_t^{\mathbb{Q}}\left[Y_{t+\Delta+j\Delta}^{(n)}\right] = \left(\Psi_j^{\mathbb{Q}(n)}\right)^\top \zeta_t, \quad \forall j \geq 0. \tag{E.8}
$$

We show that $\Psi_j^{\mathbb{Q}(n)},\ j \geq 0$ satisfies the recursion

$$
\Psi_j^{\mathbb{Q}(n)} = \left(\lambda_{1f}^{(1)} \odot \lambda_{2f}^{(j+1)}\right) \odot \left(P^{\mathbb{Q}\top}\left(\lambda_{1f}^{(j)} \odot \Psi_{j-1}^{\mathbb{Q}(n)}\right)\right) \tag{E.9}
$$

with the initial condition

$$
\Psi_0^{\mathbb{Q}(n)} \text{ is the diagonal of the matrix } \left(\lambda_{2f} e^\top\right) \odot \left(U^{\mathbb{Q}(n)} P\right). \tag{E.10}
$$

E.2 Proof of Proposition 3.1

The Markov property of the model implies that

$$
\sigma_{r,t}^2 \equiv Var_t\left[r_{t,t+1}\right] = \omega_r^\top \zeta_t.
$$

We have:

$$
\begin{aligned}
\omega_r^\top \zeta_t = Var_t\left[r_{t,t+1}\right] &= Var_t\left[\sum_{j=1}^{1/\Delta} r_{t+j\Delta}\right] \\
&= \sum_{j=1}^{1/\Delta} Var_t\left[r_{t+j\Delta}\right] + 2\sum_{j=2}^{1/\Delta}\sum_{i=1}^{j-1} Cov_t\left(r_{t+i\Delta}, r_{t+j\Delta}\right).
\end{aligned}
$$

Based on previous calculations, we have:

$$
\begin{aligned}
Var_t\left[r_{t+j\Delta}\right] &= \left(\Psi_0^{(2)}\right)^\top P^{j-1}\zeta_t - \left(\left(\Psi_0^{(1)}\right)^\top P^{j-1}\zeta_t\right)^2 \\
Cov_t\left(r_{t+i\Delta}, r_{t+j\Delta}\right) &= \left(\Psi_{j-i}^{(11)}\right)^\top P^{i-1}\zeta_t - \left(\left(\Psi_0^{(1)}\right)^\top P^{i-1}\zeta_t\right)\left(\left(\Psi_0^{(1)}\right)^\top P^{j-1}\zeta_t\right).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\omega_r = &\sum_{j=1}^{1/\Delta}\left(\left(\Psi_0^{(2)}\right)^\top P^{j-1} - \left(\left(\Psi_0^{(1)}\right)^\top P^{j-1}\right)\odot\left(\left(\Psi_0^{(1)}\right)^\top P^{j-1}\right)\right)^\top \\
&+ 2\sum_{j=2}^{1/\Delta}\sum_{i=1}^{j-1}\left(\left(\Psi_{j-i}^{(11)}\right)^\top P^{i-1} - \left(\left(\Psi_0^{(1)}\right)^\top P^{i-1}\right)\odot\left(\left(\Psi_0^{(1)}\right)^\top P^{j-1}\right)\right)^\top.
\end{aligned}
$$

The Markov property of the model implies

$$
E_t\left[\sigma_{r,t+j\Delta}^2\right] = \Upsilon_j^\top\zeta_t \quad\text{and}\quad E_t^{\mathbb{Q}}\left[\sigma_{r,t+j\Delta}^2\right] = \Upsilon_j^{\mathbb{Q}\top}\zeta_t,
$$

and we show that:

$$
\Upsilon_j = \left(\omega_r^\top P^j\right)^\top \quad\text{and}\quad \Upsilon_j^{\mathbb{Q}} = \left(\omega_r^\top P^{\mathbb{Q}(j)}\right)^\top. \tag{E.11}
$$

It follows that

$$
E_t\left[\sigma_{r,t+1}^2\right] = \Upsilon_{1/\Delta}^\top\zeta_t \quad\text{and}\quad E_t^{\mathbb{Q}}\left[\sigma_{r,t+1}^2\right] = \Upsilon_{1/\Delta}^{\mathbb{Q}\top}\zeta_t,
$$

which implies Eq. (24) in Proposition 3.1.

One can show that

$$
E_t\left[\sigma_{t,t+1}^2\right] = \left(\Psi_0^{(2)}\right)^\top\left(\sum_{j=1}^{1/\Delta}P^{j-1}\right)\zeta_t \quad\text{and}\quad E_t^{\mathbb{Q}}\left[\sigma_{t,t+1}^2\right] = \sum_{j=1}^{1/\Delta}\left(\Psi_{j-1}^{\mathbb{Q}(2)}\right)^\top\zeta_t,
$$

which leads to Eq. (25) in Proposition 3.1.

Figure 1: Model Asset Pricing Moments: GDA $\psi > 1$

The entries of the figure are the first and second moments of the log price-dividend ratio, the log risk-free rate and excess log equity returns, and finally the slope and R^2 for the regression of 5-year future excess log equity returns onto the current log price dividend ratio.

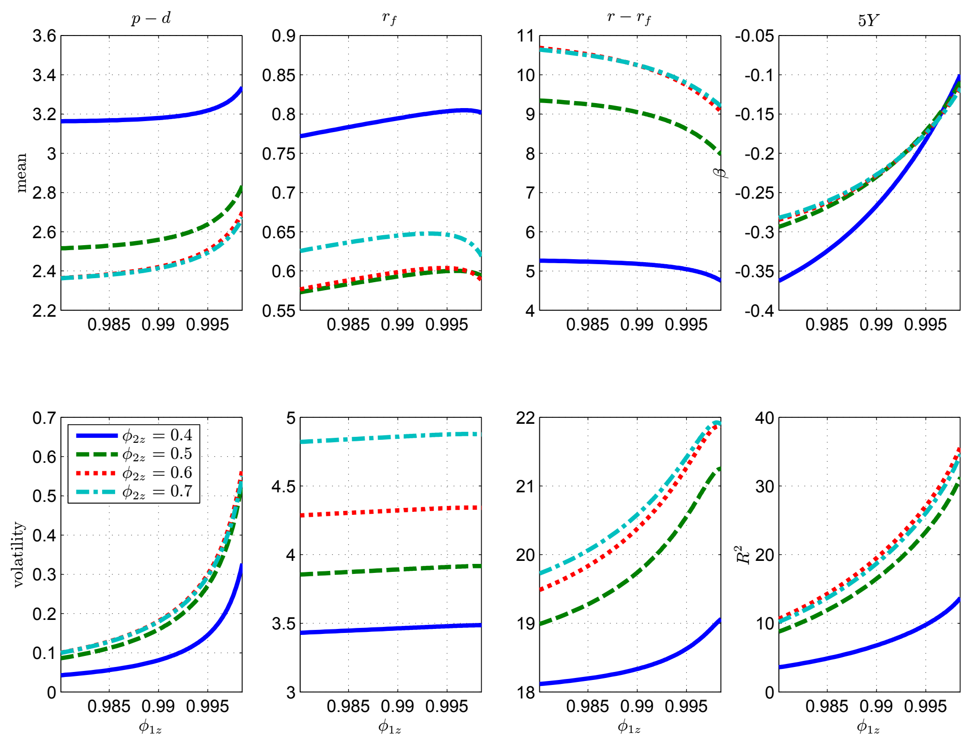


Figure 2: Model Asset Pricing Moments: GDA $\psi < 1$

The entries of the figure are the first and second moments of the log price-dividend ratio, the log risk-free rate and excess log equity returns, and finally the slope and R^2 for the regression of 5-year future excess log equity returns onto the current log price dividend ratio.

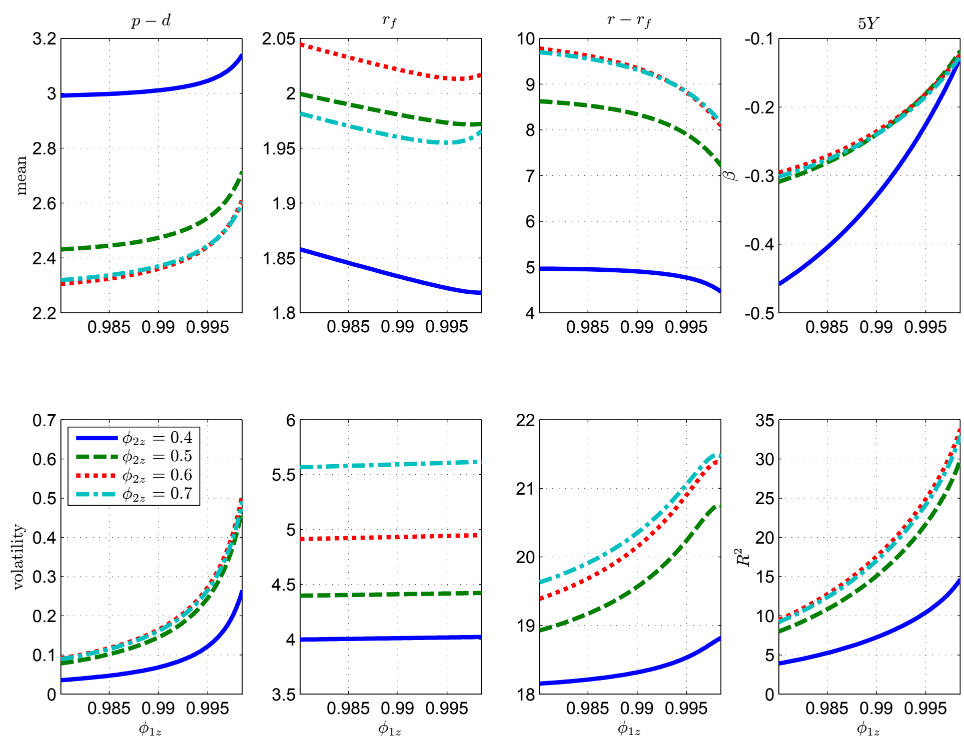


Figure 3: Model Variance Premium Moments: GDA $\psi < 1$

The entries of the table are the first and second moments of the options implied variance, the realized variance and the variance premium. All measures are on a monthly basis in percentage-squared.

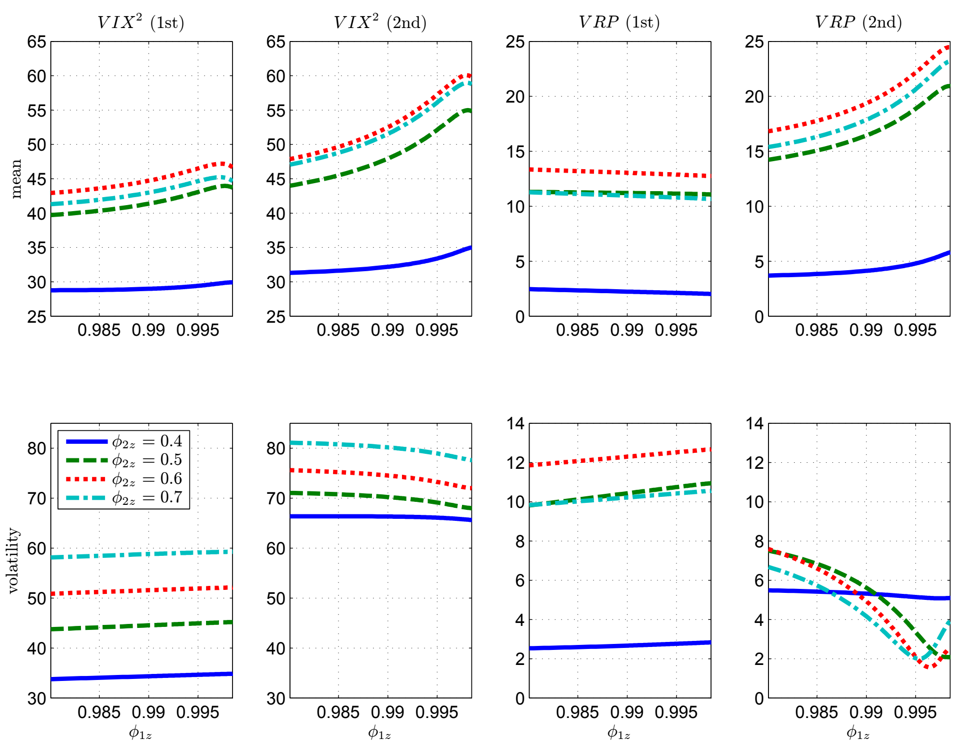


Figure 4: Model Short-Run Risk-Return Trade-Offs: GDA $\psi < 1$

The entries of the table are the slope coefficients as well as the coefficients of determination (R_l^2) of the regression

$$\frac{r_{t,t+l}}{l} = \alpha_{0l} + \beta_{1,0l} vp_t + \epsilon_{t,t+l}^{(0)}$$

where vp_t is the current monthly variance premium, and $r_{t,t+l}$ is the accumulated future monthly returns over l months.

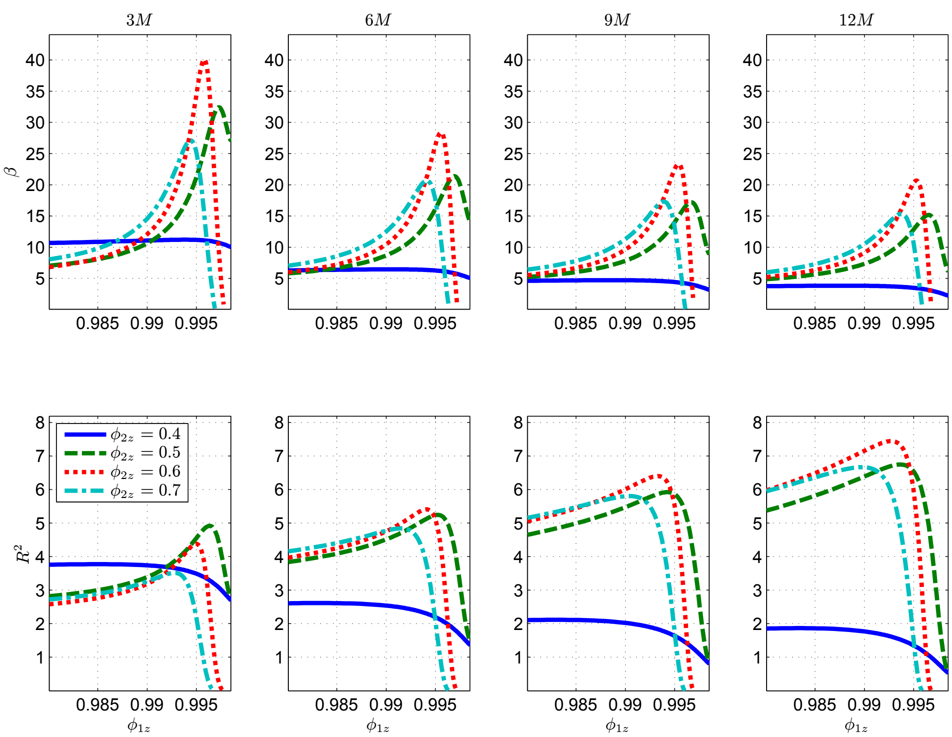
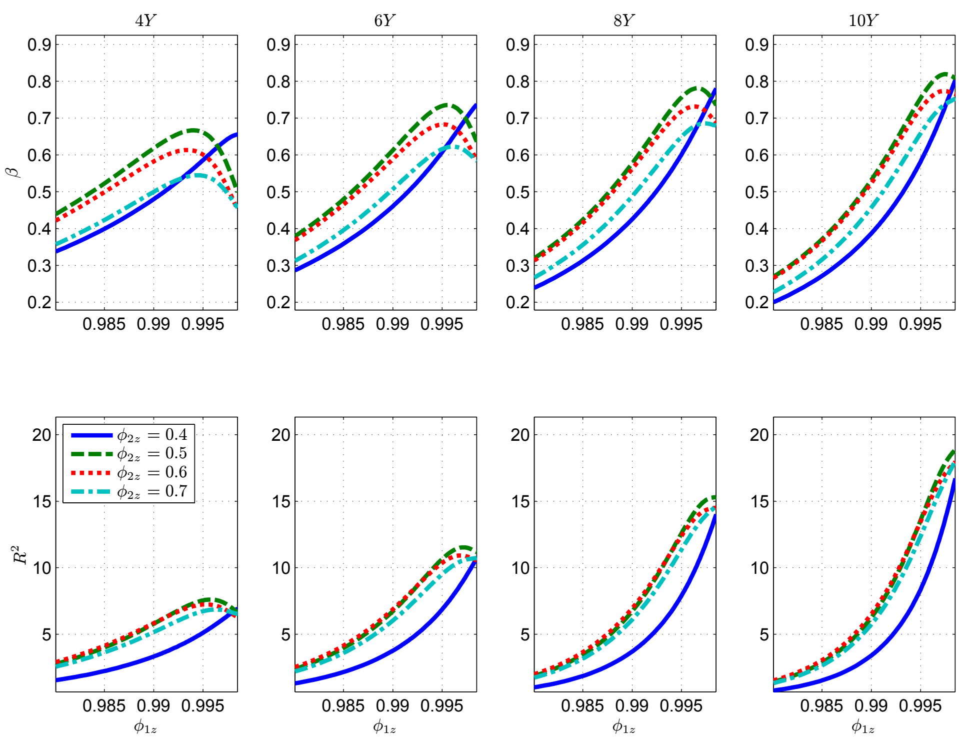


Figure 5: Model Long-Run Risk-Return Trade-Offs: GDA $\psi < 1$

The entries of the table are the slope coefficients as well as the coefficients of determination (R^2) of the regression

$$\frac{r_{t,t+h}}{h} = \alpha_{hh} + \beta_{hh}\frac{\sigma^2_{t-h,t}}{h} + \epsilon^{(h)}_{t,t+h}$$

were $\sigma^2_{t-h,t}$ is the accumulated past monthly realized variance over the last h months and $r_{t,t+h}$ is the accumulated future monthly returns over the next h months.



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