

ARMA Representation of Integrated and Realized Variances*

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First version: July 6, 2002

This version: July 6, 2003

Abstract

This paper derives the ARMA representation of integrated and realized variances when the spot variance depends linearly on two autoregressive factors, i.e., SR-SARV(2) models. This class of processes includes affine, GARCH diffusion, CEV models, as well as the eigenfunction stochastic volatility and the positive Ornstein-Uhlenbeck models. We also study the leverage effect case, the relationship between weak GARCH representation of returns and the ARMA representation of realized variances. Finally, various empirical implications of these ARMA representations are considered. We find that it is possible that some parameters of the ARMA representation are negative. Hence, the positiveness of the expected values of integrated or realized variances is not guaranteed. We also find that for some frequencies of observations, the continuous time model parameters may be weakly or not identified through the ARMA representation of realized variances.

Key words: Integrated variance, realized variance, ARMA representation, SR-SARV models, weak identification.

*The author would like to thank Torben Andersen, Tim Bollerslev, and especially Bryan Campbell, Neil Shephard and a referee for helpful discussions and comments. The author acknowledges financial support from FQRSC, IFM2, MITACS, NSERC, SSHRC, and Canada Research Chair Program (Econometrics Chair: Jean-Marie Dufour). The author thanks the Bendheim Center for Finance, Princeton University, for its hospitality during his visit and the revision of this paper, and is responsible for any remaining errors.

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1 Introduction

The recent literature on volatility modeling has highlighted the advantage of using realized variances constructed from the summation of finely-sampled squared high-frequency returns. These papers include Andersen and Bollerslev (1998), Andersen, Bollerslev, Diebold and Ebens (2001), Andersen, Bollerslev, Diebold and Labys (2001, 2003; ABDL hereafter), Barndorff-Nielsen and Shephard (2001, 2002a-c, 2003), Taylor and Xu (1997), and Zhou (1996); for a survey of this literature, Andersen, Bollerslev and Diebold (2001), Barndorff-Nielsen, Nicolato and Shephard (2002), and Dacorogna et al. (2001) should be consulted. The theoretical justification for this approach is that when the length of the intra-daily returns tends to zero, the sum of squared returns tends in probability to the quadratic variation of the underlying diffusion process (ABDL, 2001; Barndorff-Nielsen and Shephard, 2001; Comte and Renault, 1998). Quadratic variation plays a central role in the option pricing literature. In particular, when there are no jumps, quadratic variation equals the integrated variance highlighted by Hull and White (1987).

Concurrently, it has been well established by several empirical studies that two factors are needed to model correctly the spot variance process: one factor to capture the persistence of the volatility and a second one to deal with fat-tails; examples of these studies include Engle and Lee (1999), Gallant, Hsu and Tauchen (1999), Meddahi (2001), Alizadeh, Brandt and Diebold (2002), Barndorff-Nielsen and Shephard (2002a), Bollerslev and Zhou (2002), and Chernov, Gallant, Ghysels and Tauchen (2003).¹

The main goal of the paper is to derive the ARMA representation of integrated and realized variances when the spot variance depends linearly on two autoregressive factors. Continuous time stochastic volatility models where the spot variance depends linearly on autoregressive factors are studied in detail by Meddahi and Renault (2002) who named them the square-root stochastic autoregressive variance (SR-SARV) models in reference to the discrete time counterpart introduced by Andersen (1994). Special examples of SR-SARV models are: the affine model of Heston (1993); the GARCH diffusion model of Nelson (1990); the CEV processes when the variance is square-integrable (Meddahi and Renault, 2002); the positive Ornstein-Uhlenbeck model of Barndorff-Nielsen and Shephard (2001); and the eigenfunction stochastic volatility model of Meddahi (2001).

Knowing the ARMA representation of integrated and realized variances is important for impulse response analysis, filtering, forecasting, and for statistical inference purposes. For example, by using these ARMA representations, one can forecast future values of integrated or realized variances by using current and past realized variances. Indeed, this approach is adopted in Andersen, Bollerslev and Meddahi (2002) who used the results of the present paper.

¹An alternative approach to the two-factor model is to consider a one-factor model with jumps, as in Andersen, Benzoni and Lund (2002) and Pan (2002); for a comprehensive empirical comparison of these approaches, see Chernov et al. (2003).

The ARMA representation is also the analytical steady-state of integrated variance. Hence, instead of using the Kalman filter in a QML estimation procedure as did Barndorff-Nielsen and Shephard (2002a), one can use the ARMA representation. In addition, knowing the ARMA representation of the integrated variance process (and their moments derived in Andersen, Bollerslev and Meddahi, 2002) allows one to implement tests for equal or superior predictive ability² of different continuous time stochastic volatility processes, using the realized variance as a proxy for the integrated variance.

Barndorff-Nielsen and Shephard (2002a) showed that integrated and realized variances are ARMA(p,p) processes when the spot variance is a linear combination of p independent continuous time autoregressive processes. This result was extended by Andersen, Bollerslev and Meddahi (2002), who establish the same result when the variance is a linear combination of p uncorrelated and autoregressive processes. However, these studies did not characterize all the parameters of the ARMA(p,p) processes. While the autoregressive parameters of the ARMA(p,p) process coincide with those of the p autoregressive factors involved in the spot variance, the characterization of the moving-average parameters is less obvious. To do so, we use the results of Meddahi (2002a) who characterized these moving-average parameters when $p = 2$. Finally, it is worth noting that the ARMA representation of realized variance is not the same as the weak GARCH representation of returns (Drost and Nijman, 1993; Drost and Werker, 1996). However, a weak GARCH structure of intra-daily returns implies the ARMA structure of realized variances. In Section 4, we elaborate in more detail the relationship between the weak GARCH representation of returns and the ARMA representation of realized variances.

After deriving the ARMA representation of integrated and realized variances, we study their empirical implications, and find two main important results. First of all, when one writes the (GARCH-like) recursive equation of the expected value of integrated or realized variances, one can possibly get negative parameters. Hence, the positivity of the expected value of integrated or realized variances is not ensured. This result is not in contradiction with the theoretical aspects of the model: it is possible that the linear projection of a positive variable onto the Hilbert-space generated by its past values is non-positive. However, the conditional expectation of a positive variable given the sigma-algebra generated by its past values is always positive. This non-positivity problem was the main motivation of Meddahi and Renault (2002) to use SR-SARV models instead of weak GARCH ones to study temporal aggregation of volatility models. The second empirical result is that it appears that some parameters of the structural models, i.e., the parameters of the continuous time model, are weakly identified when the spot variance depends on two factors. The main reason is that one moving-average root of the realized variance process is close to an autoregressive root and, indeed, is the same for

²For predictive ability analysis, see, e.g., Diebold and Mariano (1995), West (1996), and Corradi and Swanson (2002).

a particular frequency of observations that depends on the unknown parameters. This may explain why some estimates are not precise in Bollerslev and Zhou (2002), who used GMM (Hansen, 1982) to estimate a two-factor affine continuous time model by using the dynamics of the realized variance process.

The paper is organized as follows. In Section 2, we give the structure of the autoregressive variance processes and provide several examples. We then characterize in Section 3 the ARMA representation of integrated and realized variances when the spot variance depends on one autoregressive factor, given that this has not been previously presented in the literature. Section 3 also deals with the more empirically relevant two-factor example. In Section 4, we study the case of leverage effect, the link between weak GARCH of returns and ARMA representation of realized variances, and the usefulness of the ARMA representation of realized variance for the estimation of the parameters of the continuous time stochastic volatility model by using the GMM or the QML methods. Section 5 studies the empirical implications of the ARMA representation of integrated and realized variances. Section 6 concludes, and all the proofs are reported in the Appendix.

2 Stochastic Autoregressive Variance Models

We assume that

$$dp_t = \sigma_t dW_t, \quad (2.1)$$

$$\sigma_t^2 = a_0 + aP(f_t) + \tilde{a}\tilde{P}(f_t) \quad (2.2)$$

where f_t is a state-variable process, possibly bivariate, and independent of the process W_t . The functions $P(\cdot)$ and $\tilde{P}(\cdot)$ are defined so that they have the following properties:

$$E[P(f_t)] = E[\tilde{P}(f_t)] = 0, \quad Var[P(f_t)] = Var[\tilde{P}(f_t)] = 1, \quad (2.3)$$

$$Cov[P(f_t), \tilde{P}(f_t)] = 0, \quad (2.4)$$

$$\begin{aligned} \forall h > 0, E[P(f_{t+h}) | f_t, p_t, \tau \leq t] &= \exp(-\lambda h)P(f_t), \\ E[\tilde{P}(f_{t+h}) | f_t, p_t, \tau \leq t] &= \exp(-\tilde{\lambda} h)\tilde{P}(f_t), \end{aligned} \quad (2.5)$$

where λ and $\tilde{\lambda}$ are two positive real numbers. In the first part of the following section, we will assume that \tilde{a} equals zero and we will call it the one-factor model. In the second part of the same section, we will assume that $a \neq 0$ and $\tilde{a} \neq 0$, and we will call it the two-factor model.

Equations (2.3) are normalization assumptions. Equation (2.4) means that the components $P(f_t)$ and $\tilde{P}(f_t)$ are uncorrelated. This assumption met when f_t is a bivariate process

$(f_{1,t}, f_{2,t})^\top$ where $f_{1,t}$ and $f_{2,t}$ are independent, $P(f_t)$ is a function of $f_{1,t}$, and $\tilde{P}(f_t)$ is a function of $f_{2,t}$. This independence assumption is, however, not necessary for ensuring (2.4); see the eigenfunction example below. Observe that (2.3) and (2.4) imply

$$E[\sigma_t^2] = a_0 \quad \text{and} \quad \text{Var}[\sigma_t^2] = a^2 + \tilde{a}^2. \quad (2.6)$$

Assumption (2.5) means that each component of the spot variance is an AR(1) process. As we will show below, when explicit examples will be provided, assumption (2.5) holds for the popular affine stochastic volatility models (Heston, 1993) and the GARCH diffusion model (Nelson, 1990). It holds also for the CEV processes, the eigenfunction stochastic volatility (ESV) models of Meddahi (2001), and the positive Ornstein-Uhlenbeck Lévy-driven models of Barndorff-Nielsen and Shephard (2001). Observe that under (2.5), we have

$$\forall h > 0, \quad E[\sigma_{t+h}^2 \mid f_\tau, p_\tau, \tau \leq t] = a_0 + a \exp(-\lambda h)P(f_t) + \tilde{a} \exp(-\tilde{\lambda} h)\tilde{P}(f_t). \quad (2.7)$$

This model is a special case of the SR-SARV model introduced by Andersen (1994) in discrete time and extended to continuous time by Meddahi and Renault (2002). It is a special case because Meddahi and Renault (2002) assumed that the variance is a linear combination of the components of a general VAR(1) process, while we are assuming that the VAR(1) process has a diagonal autoregressive matrix. Because we specify the independence of the factor f_t with the Brownian process W_t , we exclude the leverage effect, and accordingly the theoretical results of Meddahi and Renault (2002) imply that any discrete process $\{\varepsilon_{th}^{(h)}, \varepsilon_{th}^{(h)} \equiv p_{th} - p_{(t-1)h}, t \in N\}$, where h a positive real number, is a weak GARCH(2,2) (Drost and Nijman, 1993; Drost and Werker, 1996); see Meddahi (2002a) for the characterization of the weak GARCH parameters. We now give some examples of the model characterized by (2.1)-(2.5).

Example 1: Affine processes, Heston (1993). Assume that $\sigma_t^2 = \sigma_{1,t}^2 + \sigma_{2,t}^2$ where $\sigma_{1,t}^2$ and $\sigma_{2,t}^2$ are two independent square-root processes

$$\sigma_{i,t}^2 = k_i(\theta_i - \sigma_{i,t}^2)dt + \sigma_i \sigma_{it} dW_{i,t}, \quad i = 1, 2.$$

Then we can rewrite σ_t^2 as in (2.2)-(2.5), with $f_t = (\sigma_{1,t}^2, \sigma_{2,t}^2)^\top$,

$$P(f_t) = \frac{\sqrt{2k_1}}{\sqrt{\theta_1 \sigma_1^2}}(\theta_1 - \sigma_{1,t}^2), \quad \tilde{P}(f_t) = \frac{\sqrt{2k_2}}{\sqrt{\theta_2 \sigma_2^2}}(\theta_2 - \sigma_{2,t}^2),$$

$$a_0 = \theta_1 + \theta_2, \quad a = -\sigma_1 \sqrt{\theta_1/2k_1}, \quad \tilde{a} = -\sigma_2 \sqrt{\theta_2/2k_2}, \quad \lambda = k_1, \quad \tilde{\lambda} = k_2.$$

Example 2: GARCH diffusion processes, Nelson (1990). Assume that $\sigma_t^2 = \sigma_{1,t}^2 + \sigma_{2,t}^2$ where $\sigma_{1,t}^2$ and $\sigma_{2,t}^2$ are two independent GARCH diffusion processes

$$\sigma_{i,t}^2 = k_i(\theta_i - \sigma_{i,t}^2)dt + \sigma_i^2 \sigma_{it} dW_{i,t}, \quad \sigma_i^2 < 2k_i, \quad i = 1, 2.$$

Note that we assume that $\sigma_i^2 < 2k_i$, $i = 1, 2$, in order to ensure the existence of the second moment of $\sigma_{i,t}^2$, $i = 1, 2$, which is an assumption that one needs for the existence of the second moments of integrated and realized variances. We can also rewrite σ_t^2 as in (2.2)-(2.5), with $f_t = (\sigma_{1,t}^2, \sigma_{2,t}^2)^\top$,

$$P(f_t) = \frac{\sqrt{2k_1 - \sigma_1^2}}{\theta_1 \sigma_1} (\sigma_{1,t}^2 - \theta_1), \quad \tilde{P}(f_t) = \frac{\sqrt{2k_2 - \sigma_2^2}}{\theta_2 \sigma_2} (\sigma_{2,t}^2 - \theta_2),$$

$$a_0 = \theta_1 + \theta_2, \quad a = \theta_1 \sqrt{\sigma_1^2 / (2k_1 - \sigma_1^2)}, \quad \tilde{a} = \theta_2 \sqrt{\sigma_2^2 / (2k_2 - \sigma_2^2)}, \quad \lambda = k_1, \quad \tilde{\lambda} = k_2.$$

Example 3: Eigenfunction stochastic volatility model, Meddahi (2001). Assume that f_t is a scalar stationary diffusion process given by the stochastic differential equation

$$df_t = \mu(f_t)dt + \sigma(f_t)d\tilde{W}_t,$$

where \tilde{W}_t is a Brownian process. Let \mathcal{A} be the infinitesimal generator associated with f_t and defined by

$$\mathcal{A}\phi(f_t) \equiv \mu(f_t)\phi'(f_t) + \frac{\sigma^2(f_t)}{2}\phi''(f_t),$$

for any square-integrable and twice differentiable function, $\phi(f_t)$. The solution $P_i(f_t)$ of

$$\mathcal{A}P_i(f_t) = -\lambda_i P_i(f_t),$$

is called an eigenfunction, while $(-\lambda_i)$ is the corresponding eigenvalue.³ It turns out that the eigenfunctions satisfy the conditions (2.3), (2.4) and (2.5). Therefore, we can assume that the autoregressive processes $P(f_t)$ and $\tilde{P}(f_t)$ in (2.2) are two particular eigenfunctions. Observe that in this case, $P(f_t)$ and $\tilde{P}(f_t)$ are orthogonal but not independent.

Meddahi (2001) shows that most of the volatility models, with one factor or more, are special examples of ESV models. It is the case for affine processes where the eigenfunction are Laguerre polynomials. Thus, the first example is an ESV model. Meddahi (2001) also shows that the GARCH diffusion example and hence Example 2 is an ESV model. When the state variable f_t is an Ornstein-Uhlenbeck process, as in the log-normal model of Hull and White (1987) and Wiggins (1987), the corresponding eigenfunctions are the Hermite polynomials. Therefore, the log-normal volatility model is an ESV model, but the decomposition of the variance process σ_t^2 in terms of linear combination of Hermite polynomials involves all the infinite number of polynomials. In contrast, one can assume that σ_t^2 depends on two particular

³For a more detailed discussion of the properties of infinitesimal generators, see, e.g., Hansen and Scheinkman (1995) and Ait-Sahalia, Hansen and Scheinkman (2002); see also Chen, Hansen and Scheinkman (2000) for an alternative approach of modeling continuous time processes through eigenfunctions.

Hermite polynomials, i.e., $P(f_t)$ and $\tilde{P}(f_t)$ in (2.2) are two Hermite polynomials, $H_i(f_t)$ and $H_j(f_t)$. For instance, Meddahi (2001) studies the case

$$\sigma_t^2 = a_0 + a_1 H_1(f_t) + a_2 H_2(f_t), \quad H_1(f_t) = f_t, \quad H_2(f_t) = (f_t^2 - 1)/\sqrt{2},$$

with $a_1^2 - 4a_2/\sqrt{2}(a_0 - a_2/\sqrt{2}) \leq 0$ and $a_2 > 0$ in order to ensure the positivity of σ_t^2 . Finally, note also that Meddahi (2001) proposes other continuous time factors within the eigenfunction framework, particularly an example based on the Jacobi diffusion.

Example 4: Positive Ornstein-Uhlenbeck processes, Barndorff-Nielsen and Shephard (2001). Assume that $\sigma_t^2 = \sigma_{1,t}^2 + \sigma_{2,t}^2$ where $\sigma_{1,t}^2$ and $\sigma_{2,t}^2$ are two independent positive Ornstein-Uhlenbeck processes

$$\sigma_{i,t}^2 = e^{-k_i t} \sigma_{0,i}^2 + \int_0^t e^{-k_i(t-s)} dz_i(k_i s), \quad k_i > 0, \quad \sigma_{0,i}^2 = \int_{-\infty}^0 e^s dz(s),$$

where $z_1(t)$ and $z_2(t)$ are two independent integrable homogenous Lévy processes with positive increments. In addition, assume that the mean and variance of $\sigma_{i,t}^2$, $i=1,2$, exist and are denoted respectively by θ_i and v_i . Then we can rewrite σ_t^2 as in (2.2)-(2.5), with $f_t = (\sigma_{1,t}^2, \sigma_{2,t}^2)^\top$,

$$P(f_t) = \frac{\sigma_{1,t}^2 - \theta_1}{\sqrt{v_1}}, \quad \tilde{P}(f_t) = \frac{\sigma_{2,t}^2 - \theta_2}{\sqrt{v_2}},$$

$$a_0 = \theta_1 + \theta_2, \quad a = v_1, \quad \tilde{a} = v_2, \quad \lambda = k_1, \quad \tilde{\lambda} = k_2.$$

Example 5: Combination of two different factor structures. In the first, second and fourth previous examples, we always considered the same structure for the factors. For instance, in the first example, we assume that the variance is the sum of two square-root processes. However, one can also combine two different structures for the component of σ_t^2 . A simple example is to assume that σ_t^2 is the sum of a square-root process and a GARCH diffusion process. This approach, implicit in the multifactor ESV model of Meddahi (2001), is not very common in the literature; see, however, Chernov et al. (2003) for stock price dynamics modeling, and Ahn, Dittmar, Gallant and Gao (2003) for interest rate modeling.

3 ARMA Representation of Variances

3.1 Integrated and Realized Variances

In the rest of the paper, we will study the ARMA representation of the daily integrated variance and the realized variance. These two variables are defined respectively by

$$IV_t = \int_{t-1}^t \sigma_u^2 du, \tag{3.1}$$

and

$$RV_t(h) \equiv \sum_{i=1}^{1/h} \varepsilon_{t-1+ih}^{(h)2}, \quad (3.2)$$

where h is a real number such that $1/h$ is an integer, and $\varepsilon_{t-1+ih}^{(h)}$ are the intra-daily returns over the periods $[t-1+(i-1)h; t-1+ih]$, for $i = 1, 2, \dots, 1/h$. It is well-known, using the theory of quadratic variation, that $RV_t(h)$ converges in probability towards IV_t when $h \rightarrow 0$. In addition, Barndorff-Nielsen and Shephard (2002a) provide a Central Limit Theorem. Finally, for a given h , Barndorff-Nielsen and Shephard (2002a) and Meddahi (2002b) studied theoretically the difference between realized and integrated variances.⁴ Indeed, Meddahi (2002b) shows that when there is no drift, we have

$$RV_t(h) = IV_t + e_t(h), \quad (3.3)$$

where

$$e_t(h) = 2 \sum_{i=1}^{1/h} \int_{t-1+(i-1)h}^{t-1+ih} \left(\int_{t-1+(i-1)h}^u \sigma_s dW_s \right) \sigma_u dW_u.$$

Note that the convergence of realized variance towards integrated variance, plus some uniform integrability conditions, imply also that at the limit the ARMA representation of integrated and realized variances coincide. We will discuss this point below.

Before characterizing the ARMA representation of variances in models with one or two factors, let us make two remarks about these ARMA representations for a general SR-SARV(p) process. Meddahi and Renault (2002) show that for a SR-SARV(p) process, the spot variance follows an ARMA(p,p-1). Therefore, the integrated variance which is obtained by temporally aggregating the spot variance process is an ARMA(p,p) (see Granger and Morris, 1976). When there is no leverage effect and no drift, Barndorff-Nielsen and Shephard (2002a) and Meddahi (2002) show that the process $e_t(h)$ is uncorrelated with the process IV_t . Hence, realized variance is also an ARMA(p,p) and has the same autoregressive roots as integrated variance. However, the moving-average roots are different. Second, Barndorff-Nielsen and Shephard (2002a) formally showed that both integrated and realized variances follow an ARMA(p,p) when the spot variance equals the sum of p independent and autoregressive processes. Andersen, Bollerslev and Meddahi (2002) extended these results to the case where the spot variance equals the sum of p uncorrelated and autoregressive processes, as in our setting.⁵

⁴Andreu and Ghysels (2002) and Bai, Russell and Tiao (2001) also studied this difference through simulations. In particular, they take into account microstructure effects that we ignore in our study.

⁵In their proof, Bollerslev and Zhou (2002) explicitly recognized that integrated and realized variances are ARMA(p,p) processes, $p = 1, 2$, when the spot variance depends on p square-root processes.

In the derivation of the ARMA representation of integrated and realized variances, we will need some variances and covariances of these two variables. They are given by⁶

$$\text{Var}[IV_t] = 2\frac{a^2}{\lambda^2}[\exp(-\lambda) + \lambda - 1] + 2\frac{\tilde{a}^2}{\tilde{\lambda}^2}[\exp(-\tilde{\lambda}) + \tilde{\lambda} - 1], \quad (3.4)$$

$$\text{Cov}(IV_t, IV_{t-1}) = a^2\frac{[1 - \exp(-\lambda)]^2}{\lambda^2} + \tilde{a}^2\frac{[1 - \exp(-\tilde{\lambda})]^2}{\tilde{\lambda}^2}, \quad (3.5)$$

$$\text{Cov}(IV_t, IV_{t-2}) = a^2\exp(-\lambda)\frac{[1 - \exp(-\lambda)]^2}{\lambda^2} + \tilde{a}^2\exp(-\tilde{\lambda})\frac{[1 - \exp(-\tilde{\lambda})]^2}{\tilde{\lambda}^2}, \quad (3.6)$$

$$\text{Var}[RV_t(h)] = \text{Var}[IV_t] + \text{Var}[e_t(h)], \quad \text{with}$$

$$\text{Var}[e_t(h)] = 2a_0^2h + \frac{4a^2}{h\lambda^2}(\exp(-\lambda h) - 1 + \lambda h) + \frac{4\tilde{a}^2}{h\tilde{\lambda}^2}(\exp(-\tilde{\lambda}h) - 1 + \tilde{\lambda}h), \quad (3.7)$$

$$\text{Cov}(RV_t(h), RV_{t-1}(h)) = \text{Cov}(IV_t, IV_{t-1}), \quad (3.8)$$

$$\text{Cov}(RV_t(h), RV_{t-2}(h)) = \text{Cov}(IV_t, IV_{t-2}). \quad (3.9)$$

In the subsequent propositions, we will also use the following notation. Let z_t be second-order stationary variable, then we denote by

$$m_{t-1}[z] \quad (3.10)$$

the best linear predictor of z_t onto $H_{t-1}(z)$, where $H_{t-1}(z)$ is the Hilbert-space generated by $\{1, z_\tau, \tau \leq t-1\}$. For real numbers $\gamma_1, \gamma_2, v_0, v_1, v_2, \rho_1$, and ρ_2 with $\rho_2 \neq 0$, we define the following functions:

$$D_{10}(\gamma, v_0, v_1) = (1 + \gamma^2)v_0 - 2\gamma v_1, \quad (3.11)$$

$$D_{11}(\gamma, v_0, v_1) = -\gamma v_0 + v_1, \quad (3.12)$$

$$D_{20}(\gamma_1, \gamma_2, v_0, v_1, v_2) = (1 + \gamma_1^2\gamma_2^2 + (\gamma_1 + \gamma_2)^2)v_0 - 2(\gamma_1 + \gamma_2)(1 + \gamma_1\gamma_2)v_1 + 2\gamma_1\gamma_2v_2, \quad (3.13)$$

$$D_{21}(\gamma_1, \gamma_2, v_0, v_1, v_2) = -(1 + \gamma_1\gamma_2)(\gamma_1 + \gamma_2)v_0 + (1 + (\gamma_1 + \gamma_2)^2 + \gamma_1\gamma_2)v_1 - (\gamma_1 + \gamma_2)v_2, \quad (3.14)$$

⁶Barndorff-Nielsen and Shephard (2002a) give the formulas (3.4)-(3.9) when the spot variance depends on independent and autoregressive processes. By using the results of Meddahi (2001, 2002b), the independence assumption was relaxed by Andersen, Bollerslev and Meddahi (2002), who assumed that the autoregressive processes are uncorrelated.

$$D_{22}(\gamma_1, \gamma_2, v_0, v_1, v_2) = \gamma_1 \gamma_2 v_0 - (\gamma_1 + \gamma_2) v_1 + v_2, \quad (3.15)$$

$$S(\rho_1, \rho_2) = 2^{-1} \rho_2^2 \rho_1^{-2} \left[-2 - \rho_2^{-1} + \text{sign}(\rho_2) \sqrt{(2 + \rho_2^{-1})^2 - 4\rho_1^2 \rho_2^{-2}} \right], \quad (3.16)$$

where

$$\text{sign}(\rho_2) = 1 \text{ if } \rho_2 > 0 \text{ while } \text{sign}(\rho_2) = -1 \text{ if } \rho_2 < 0.$$

We are now able to study the ARMA dynamics of integrated and realized variances. We study the one-factor case in the next subsection. Then we study the more empirically relevant two-factor example in the last subsection.

3.2 The One-Factor Model

Throughout this subsection, we assume that the variance process depends on one factor, i.e., \tilde{a} in (2.2) equals zero. We start by the derivation of the ARMA(1,1) representation of the integrated variance process.

Proposition 3.1 ARMA(1,1) representation of integrated variance for the one-factor model. *Consider the model defined by (2.1)-(2.5) with $a \neq 0$ and $\tilde{a} = 0$. Then IV_t , the integrated variance defined in (3.1), is an ARMA(1,1) process with the following representation:*

$$IV_t = (1 - \gamma)a_0 + \gamma IV_{t-1} + \eta_t - \beta \eta_{t-1}, \quad (3.17)$$

where η_t is a white noise (whose variance is given in the Appendix), with

$$\gamma = \exp(-\lambda), \quad \beta = \frac{-1 + \sqrt{1 - 4\rho^2}}{2\rho}, \quad (3.18)$$

$$\rho = \frac{D_1}{D_0}, \quad D_j = D_{1j}(\gamma, \text{Var}[IV_t], \text{Cov}[IV_t, IV_{t-1}]), \quad j = 0, 1,$$

where $D_{10}(\cdot)$ and $D_{11}(\cdot)$ are defined in (3.11) and (3.12) respectively, while $\text{Var}[IV_t]$ and $\text{Cov}(IV_t, IV_{t-1})$ are given in (3.4) and (3.5) (with $\tilde{a} = 0$). As a consequence, $m_{t-1}[IV]$ (defined in (3.10)) follows the recursive formula

$$m_{t-1}[IV] = \omega + \alpha IV_{t-1} + \beta m_{t-2}[IV], \quad (3.19)$$

where $\omega = (1 - \gamma)a_0$ and $\alpha = \gamma - \beta$.

The proof of the proposition (provided in the Appendix) has two steps. In the first, we characterize the state-space representation of the integrated variance process IV_t by showing that

$$IV_t = s_{t-1} + u_t, \quad s_t = a_0(1 - \exp(-\lambda)) + \exp(-\lambda)s_{t-1} + v_t, \quad (3.20)$$

where s_t is an affine function of σ_t^2 and $(u_t, v_t)^\top$ is a martingale difference sequence. This representation is useful because it shows that IV_t is an ARMA(1,1) given that it equals an AR(1) process, s_{t-1} , plus a noise, u_t . In the second step, we also use (3.20) to characterize the ARMA coefficients: (3.20) implies that the autoregressive root of IV_t is $\exp(-\lambda)$, and the process z_t defined by

$$z_t \equiv IV_t - \exp(-\lambda)IV_{t-1} - a_0(1 - \exp(-\lambda)) = u_t - \exp(-\lambda)u_{t-1} + v_{t-1}$$

is a MA(1) process. Hence, z_t may be represented as

$$z_t = \eta_t - \beta\eta_{t-1}$$

where η_t is a white noise and the moving-average root β is a real number with $|\beta| < 1$. The real β is then obtained as the solution (with absolute value smaller than one) of

$$-\frac{\beta}{1 + \beta^2} = \frac{Cov(z_t, z_{t-1})}{Var[z_t]}.$$

It turns out that the real ρ , defined in Proposition 3.1, is exactly the ratio $Cov(z_t, z_{t-1})/Var[z_t]$, and it can be expressed in terms of $Var[IV_t]$ and $Cov(IV_t, IV_{t-1})$, as done in Proposition 3.1.

It is worth noting that, in general, the process η_t is not a martingale difference sequence. The main reason is that the process z_t is generally heteroskedastic. Therefore, η_t is not Gaussian and indeed heteroskedastic; see Meddahi and Renault (2002) and Meddahi (2002b) for more details. However, these authors showed that the process η_t is more restricted than a white noise, given that the following condition holds

$$E[\eta_t - \beta\eta_{t-1} \mid \eta_\tau, \tau \leq t-2] = 0.$$

Such multi-period conditional moment restrictions were introduced by Hansen (1985) and studied in detail by Hansen, Heaton and Ogaki (1987), Hansen and Singleton (1986), West (2001), and Kuersteiner (2002). They are derived in the context of squared residuals in Meddahi and Renault (2002) and applied in the context of integrated variance by Bollerslev and Zhou (2002); see Section 4 for more details.

The recursive equation (3.19) followed by $m_{t-1}[IV]$ is easily obtained from (3.1) given that $m_{t-1}[IV] = IV_t - \eta_t$. The most interesting feature of (3.19) is that it resembles a GARCH(1,1) equation (Bollerslev, 1986). Hence, it can be used for statistical inference, for filtering and forecasting purposes as in Baillie and Bollerslev (1992); see Andersen, Bollerslev and Meddahi (2002) for an application.

A natural question concerns the positiveness of $m_{t-1}[IV]$. This positiveness is guaranteed when $\alpha \geq 0$ and $\beta \geq 0$ (Nelson and Cao, 1992), that is,

$$0 \leq \beta \leq \exp(-\lambda).$$

It turns out that the previous conditions are not always satisfied. In particular, by using the definition of the function $D_{11}(\cdot)$ given in (3.12), it is easy to show that

$$0 \leq \beta \iff \frac{Cov(IV_t, IV_{t-1})}{Var[IV_t]} \leq \exp(-\lambda),$$

which is not satisfied. The intuition for the existence of a situation where the last inequality is not satisfied is the following. The integrated process IV_t is the temporal aggregation of the spot variance. In practice, spot variance is persistent. Thus, the aggregation of the spot variance will lead to a process with more persistence (measured by the first autocorrelation). For instance, we have

$$\frac{Cov(IV_t, IV_{t-1})}{Var[IV_t]} = 0.967 \quad \text{when} \quad \exp(-\lambda) = 0.95$$

and

$$\frac{Cov(IV_t, IV_{t-1})}{Var[IV_t]} = 0.569 \quad \text{when} \quad \exp(-\lambda) = 0.4.$$

Indeed, the inequality is always violated because when using formulas (3.4) and (3.5) (with $\tilde{a} = 0$), one gets

$$0 \leq \beta \iff 0 \leq \exp(-2\lambda) + 2\lambda \exp(-\lambda) - 1,$$

which never holds when $\lambda > 0$.⁷

In summary, the traditional assumptions for the GARCH literature that ensures the positiveness of the expected value of a positive random variable are not satisfied. We do not know if there are examples where $m_{t-1}[IV]$ is negative. However, there are no theoretical reasons that ensure the positivity of $m_{t-1}[IV]$, which is the best linear projection of IV_t onto the Hilbert-space $H_{t-1}(IV)$. However, the conditional expectation of IV_t given the sigma-algebra $\sigma(IV_\tau, \tau \leq t-1)$ is positive. The problem of the possibility of the non-positiveness of the best linear predictor of a positive variable was the main motivation of Meddahi and Renault (2002), who advocated the use of SR-SARV models instead of weak GARCH models (Drost and Nijman, 1993; Drost and Werker, 1996).

We now characterize the ARMA(1,1) representation of the realized variance process.

Proposition 3.2 ARMA(1,1) representation of realized variance for the one-factor model. *Consider the model defined by (2.1)-(2.5) with $a \neq 0$ and $\tilde{a} = 0$. Then $RV_t(h)$, the realized variance process defined in (3.2), is an ARMA(1,1) process with the representation:*

$$RV_t(h) = (1 - \gamma(h))a_0 + \gamma(h)RV_{t-1}(h) + \eta_t(h) - \beta(h)\eta_{t-1}(h), \quad (3.21)$$

⁷After the first version of this paper, Barndorff-Nielsen, Nielsen, Shephard and Ysui (2002) also considered the positiveness of estimates of integrated variance. In particular, they relate this positiveness to the positivity of the partial autocorrelation of integrated variance. It is straightforward to show that the second partial autocorrelation of the integrated variance is $\rho(\gamma - \rho)/\sqrt{(1 - \rho^2)}$, where γ and ρ are defined in Proposition 3.1, i.e., $\gamma = \exp(-\lambda)$ and ρ equals the first autocorrelation of the integrated variance process. It turns out that $\rho > \gamma$ and therefore the second partial autocorrelation of the integrated variance process is negative.

where $\eta_t(h)$ is a white noise (whose variance is given in the Appendix), with

$$\gamma(h) = \exp(-\lambda) = \gamma, \quad \beta(h) = \frac{-1 + \sqrt{1 - 4\rho(h)^2}}{2\rho(h)}, \quad (3.22)$$

$$\rho(h) = \frac{\text{Cov}(RV_t(h), RV_{t-1}(h))}{\text{Var}[RV_t(h)]}, \quad D_j(h) = D_{1j}(\gamma, \text{Var}[RV_t(h)], \text{Cov}(RV_t(h), RV_{t-1}(h))), \quad j = 0, 1.$$

Here $D_{10}(\cdot)$ and $D_{11}(\cdot)$ are defined in (3.11) and (3.12) respectively, while $\text{Var}[RV_t(h)]$ and $\text{Cov}(RV_t(h), RV_{t-1}(h))$ are given in (3.7) and (3.8) (with $\tilde{a} = 0$). As a consequence, $m_{t-1}[RV(h)]$ (defined in (3.10)) follows the recursive formula

$$m_{t-1}[RV(h)] = \omega + \alpha(h)RV_{t-1}(h) + \beta(h)m_{t-2}[RV(h)], \quad (3.23)$$

where and $\omega = (1 - \gamma)a_0$ and $\alpha(h) = \gamma - \beta(h)$.

The proof of the previous proposition is similar that the one of Proposition 3.2. Indeed, the structural representation of $RV_t(h)$ is obtained from the integrated variance structural representation by using (3.3). It turns out that the process $e_t(h)$ in (3.3) is uncorrelated with IV_t and the processes involved in the structural representation. Therefore, $RV_t(h)$ is also an ARMA(1,1) process with the same autoregressive roots than IV_t . However, their moving-average roots, $\beta(h)$ and β , are different. Indeed, we have

$$-\frac{\beta}{1 + \beta^2} = \frac{-\gamma \text{Var}[IV_t] + \text{Cov}(IV_t, IV_{t-1})}{(1 + \gamma^2) \text{Var}[IV_t] - 2\gamma \text{Cov}(IV_t, IV_{t-1})}$$

while

$$-\frac{\beta(h)}{1 + \beta(h)^2} = \frac{-\gamma(\text{Var}[IV_t] + \text{Var}[e_t(h)]) + \text{Cov}(IV_t, IV_{t-1})}{(1 + \gamma^2)(\text{Var}[IV_t] + \text{Var}[e_t(h)]) - 2\gamma \text{Cov}(IV_t, IV_{t-1})}.$$

In other words, the difference between β and $\beta(h)$ is due to the variance of the measurement error $e_t(h)$. As we already mentioned, the variance of this noise converges toward zero when h goes to zero. Hence, the moving-average root of the realized variance, as well as the variance of the innovation $\eta_t(h)$, converge to those of the integrated variance and hence the ARMA representations of integrated and realized variances coincide. This result is not surprising, given the convergence of realized variance toward integrated variance and the uniform integrability of the second moment of realized variance.

The convergence result implies that $\beta(h)$ is also negative when h is close to zero. However, it is not the case for large values of h . The main reason is as follows. Again, we have

$$0 \leq \beta(h) \iff \frac{\text{Cov}(RV_t(h), RV_{t-1}(h))}{\text{Var}[RV_t(h)]} = \frac{\text{Cov}(IV_t, IV_{t-1})}{\text{Var}[IV_t] + \text{Var}[e_t(h)]} \leq \exp(-\lambda).$$

The last inequality may hold because of the variance of the measurement error $e_t(h)$. As we will see in the empirical section, this is the case when one computes realized variance with 5-minute (or longer) returns. Thus, the linear expected value of realized variance is positive for relevant empirical examples.

3.3 The Two-Factor Model

We now study the ARMA representation of the variances when the spot variance depends on two autoregressive factors, i.e., a and \tilde{a} in (2.2) are non-zero.

Proposition 3.3 ARMA(2,2) representation of integrated variance for the two-factor model. *Consider the model defined by (2.1)-(2.5) with $a \neq 0$ and $\tilde{a} \neq 0$. Then IV_t , the integrated variance defined in (3.1), is an ARMA(2,2) process with the following representation:*

$$IV_t = (1 - \gamma_1)(1 - \gamma_2)a_0 + (\gamma_1 + \gamma_2)IV_{t-1} - \gamma_1\gamma_2IV_{t-2} + \eta_t - \beta_1\eta_{t-1} - \beta_2\eta_{t-2}, \quad (3.24)$$

where η_t is a weak white noise (whose variance is given in the Appendix), with

$$\gamma_1 = \exp(-\lambda), \quad \gamma_2 = \exp(-\tilde{\lambda}), \quad \beta_1 = \frac{\beta_2 \rho_1}{1 - \beta_2 \rho_2}, \quad \beta_2 = \frac{2s + 1 - \sqrt{4s + 1}}{2s}, \quad (3.25)$$

$$\rho_1 = \frac{D_1}{D_0}, \quad \rho_2 = \frac{D_2}{D_0}, \quad s = S(\rho_1, \rho_2)$$

$$D_j = D_{2j}(\gamma_1, \gamma_2, \text{Var}[IV_t], \text{Cov}[IV_t, IV_{t-1}], \text{Cov}[IV_t, IV_{t-2}]), \quad j = 0, 1, 2.$$

Here $D_{20}(\cdot)$, $D_{21}(\cdot)$, $D_{22}(\cdot)$ and $S(\cdot)$ are the functions defined in (3.13), (3.14), (3.15) and (3.16) respectively, while $\text{Var}[IV_t]$, $\text{Cov}(IV_t, IV_{t-1})$ and $\text{Cov}(IV_t, IV_{t-2})$ are given in (3.4), (3.5) and (3.6). As a consequence, $m_{t-1}[IV]$ (defined in (3.10)) follows the recursive formula

$$m_{t-1}[IV] = \omega + \alpha_1 IV_{t-1} + \alpha_2 IV_{t-2} + \beta_1 m_{t-2}[IV] + \beta_2 m_{t-3}[IV], \quad (3.26)$$

where $\omega = a_0(1 - \gamma_1)(1 - \gamma_2)$, $\alpha_1 = \gamma_1 + \gamma_2 - \beta_1$ and $\alpha_2 = -\gamma_1\gamma_2 - \beta_2$.

The proof of this proposition (provided in the Appendix) is similar to the one of Proposition 3.3. In particular, we considered the process z_t defined by

$$z_t \equiv IV_t - (\exp(-\lambda) + \exp(-\tilde{\lambda}))IV_{t-1} + \exp(-\lambda - \tilde{\lambda})IV_{t-2} - a_0(1 - \exp(-\lambda))(1 - \exp(-\tilde{\lambda}))$$

which, in turn, is a moving-average process of order 2, MA(2). The main difficulty is the characterization of the moving-average parameters β_1 and β_2 of the process z_t . For this purpose, we use the recent results of Meddahi (2002b), who characterized them in terms of the first and second autocorrelations of z_t . Note also that one can easily get the moving-average roots, denoted by λ_1 and λ_2 , from β_1 and β_2 . Indeed, we have

$$\beta_1 = \lambda_1 + \lambda_2 \quad \text{and} \quad \beta_2 = -\lambda_1\lambda_2,$$

and hence

$$\lambda_1 = \frac{\beta_1 - \sqrt{\beta_1^2 + 4\beta_2}}{2} \quad \text{and} \quad \lambda_2 = \frac{\beta_1 + \sqrt{\beta_1^2 + 4\beta_2}}{2}. \quad (3.27)$$

Similarly, we can also characterize the ARMA(2,2) representation of the realized variance process:

Proposition 3.4 ARMA(2,2) representation of realized variance for the two-factor model. Consider the model defined by (2.1)-(2.5) with $a \neq 0$ and $\tilde{a} \neq 0$. Then $RV_t(h)$, the realized variance process defined in (3.2), is an ARMA(2,2) process with the representation:

$$RV_t(h) = (1 - \gamma_1(h))(1 - \gamma_2(h))a_0 + (\gamma_1(h) + \gamma_2(h))RV_{t-1}(h) - \gamma_1(h)\gamma_2(h)RV_{t-2}(h) + \eta_t(h) - \beta_1(h)\eta_{t-1}(h) - \beta_2(h)\eta_{t-2}(h), \quad (3.28)$$

where $\eta_t(h)$ is a weak white noise (whose variance is given in the Appendix), with

$$\begin{aligned} \gamma_1(h) &= \exp(-\lambda) = \gamma_1, \quad \gamma_2(h) = \exp(-\tilde{\lambda}) = \gamma_2, \\ \beta_1(h) &= \frac{\beta_2(h)}{1 - \beta_2(h)} \frac{\rho_1(h)}{\rho_2(h)}, \quad \beta_2(h) = \frac{2s(h) + 1 - \sqrt{4s(h) + 1}}{2s(h)}, \end{aligned} \quad (3.29)$$

$$\rho_1(h) = \frac{D_1(h)}{D_0(h)}, \quad \rho_2(h) = \frac{D_2(h)}{D_0(h)}, \quad s(h) = S(\rho_1(h), \rho_2(h))$$

$D_j(h) = D_{2j}(\gamma_1(h), \gamma_2(h), Var[RV_t(h)], Cov[RV_t(h), RV_{t-1}(h)], Cov[RV_t(h), RV_{t-2}(h)])$, $j = 0, 1, 2$.

Here $D_{20}(\cdot)$, $D_{21}(\cdot)$, $D_{22}(\cdot)$ and $S(\cdot)$ are defined in (3.13), (3.14), (3.15) and (3.16) respectively, while $Var[RV_t(h)]$, $Cov(RV_t(h), RV_{t-1}(h))$ and $Cov(RV_t(h), RV_{t-2}(h))$ are given in (3.7), (3.8) and (3.9). As a consequence, $m_{t-1}[RV(h)]$ (defined in (3.10)) follows the recursive formula

$$\begin{aligned} m_{t-1}[RV(h)] &= \omega + \alpha_1(h)RV_{t-1}(h) + \alpha_2(h)RV_{t-2}(h) \\ &\quad + \beta_1(h)m_{t-2}[RV(h)] + \beta_2(h)m_{t-3}[RV(h)] \end{aligned} \quad (3.30)$$

where $\omega = a_0(1 - \gamma_1)(1 - \gamma_2)$, $\alpha_1(h) = \gamma_1 + \gamma_2 - \beta_1(h)$ and $\alpha_2(h) = -\gamma_1\gamma_2 - \beta_2(h)$.

The comments made after Proposition 3.2 are also relevant for the two-factor case. In addition, the moving-average roots, denoted by $\lambda_1(h)$ and $\lambda_2(h)$, are given by

$$\lambda_1(h) = \frac{\beta_1(h) - \sqrt{\beta_1(h)^2 + 4\beta_2(h)}}{2} \quad \text{and} \quad \lambda_2(h) = \frac{\beta_1(h) + \sqrt{\beta_1(h)^2 + 4\beta_2(h)}}{2}. \quad (3.31)$$

We will consider in more detail the behavior of the moving-average roots in the empirical section. As we will see, for a particular frequency of observations, one moving-average root will equal one autoregressive root. Therefore, the realized variance process will become an ARMA(1,1) process. This may be problematic for inference purposes because one faces an identification problem.

4 Refinements

4.1 The Leverage Effect Case

In the previous section, we ruled out the leverage effect; that is, we assumed that the factors driving the volatility are independent with the Brownian motion driving the process p_t .

However, we can readily extend our results to include this case. It is worth noting that the results about the ARMA representation of integrated variance do not depend on the presence of the leverage effect. In other words, they are still valid. The general results of the second and fourth propositions where we characterize the ARMA representation of the realized variance process in the one-factor and two-factor cases are still valid. The small difference is that the formula for $Var[RV_t(h)]$ given in (3.7) is no longer true.⁸ It turns out that under leverage effect, $Cov(e_t(h), IV_t) \neq 0$ and hence (3.7) becomes

$$Var[RV_t(h)] = Var[IV_t] + Var[e_t(h)] + 2Cov(e_t(h), IV_t).$$

Meddahi (2002a) gives the formulas for $Cov(e_t(h), IV_t)$ and $Var[e_t(h)]$ under leverage effect for general ESV models. Hence, the ARMA representations of realized variances follow on applying these results to Propositions 3.2 and 3.4 above.

4.2 Realized Variances and Weak GARCH

There is a connection between the ARMA representation of realized variance and weak GARCH representation of returns. Meddahi and Renault (2002) show that when one considers a continuous time SR-SARV(p) model without leverage effect, the returns are weak GARCH(p,p), whatever the frequency of observations. In other words, the squared-returns process is an ARMA(p,p). Consider now daily realized variance, defined as the sum of the square of intra-daily returns; i.e., realized variance is the temporal aggregation of the square of intra-daily returns, which are ARMA(p,p). Hence, by using Granger and Morris (1976), one obtains that daily realized variance is an ARMA(p,p) process.

However, the autoregressive and moving-average roots of the squared intra-daily returns and realized variance are not the same. For instance, for the two-factor model considered previously, the autoregressive roots of the squared intra-daily returns, $\varepsilon_{th}^{(h)}$, are $\exp(-k_1 h)$ and $\exp(-k_2 h)$, while those of the realized variance are $\exp(-k_1)$ and $\exp(-k_2)$. Of course, when $h = 1$, i.e., when one considers squared daily returns, these autoregressive roots coincide. It is not the case for the moving-average roots if one considers realized variance computed with intra-daily returns, i.e., $RV_t(\tilde{h})$, $\tilde{h} \neq 1$. Finally, when $h = \tilde{h} = 1$, realized variance equals squared daily returns and their ARMA representations are obviously the same.

4.3 GMM and QML Estimation of Continuous Time Models Through Realized Variances

For a simple exposition, we consider the one-factor case only. We have showed that the innovation process η_t given in Proposition 3.1 is such that

$$E[\eta_t - \beta\eta_{t-1} \mid \eta_\tau, \tau \leq t-2] = 0.$$

⁸Under leverage effect and constant drift, we still have $Cov(RV_t(h), RV_{t-i}(h)) = Cov(IV_t, IV_{t-i})$ for $i \neq 0$.

Thus, we have

$$E[IV_t - \exp(-\lambda)IV_{t-1} - a_0(1 - \exp(-\lambda)) \mid IV_\tau, \tau \leq t - 2] = 0. \quad (4.1)$$

Hence, if one observes the integrated variance process, one can use (4.1) to estimate a_0 and λ . However, we also have

$$E[RV_t(h) - \exp(-\lambda)RV_{t-1}(h) - a_0(1 - \exp(-\lambda)) \mid RV_\tau(h), \tau \leq t - 2] = 0, \quad (4.2)$$

which identifies (a_0, λ) regardless of the frequency of data and, therefore, provides consistent estimators free of measurement errors, $u_t(h)$. Bollerslev and Zhou (2002) follow this strategy. Observe that this consistency remains in the case of leverage effect given that (4.2) holds.

Note however that conditions (4.1) and (4.2) do not identify the parameter a . Therefore, Bollerslev and Zhou (2002) also derive a similar moment condition than (4.1) for IV_t^2 when the variance depends on square-root processes. In this case, the square of the integrated variance process also admits an ARMA representation. The reason is the following: for a square-root process f_t , Laguerre polynomials are autoregressive processes given that they are the eigenfunctions of the infinitesimal generator associated with the square-root process; see, for instance, Meddahi (2001). But any Laguerre polynomial of order i is a polynomial of degree i . Therefore, for any integer n , the vector $(f_t, f_t^2, \dots, f_t^n)^\top$ is a VAR(1). By applying this result to $n = 2$ and by using Ito's Lemma, one easily shows that IV_t^2 admits a state-space representation with $2p$ autoregressive processes, where p is the number of autoregressive processes involved in the variance decomposition. However, this result is not always true. A necessary condition is that the infinitesimal generator admits as eigenfunctions affine and quadratic functions. Wong (1964) shows that this only holds for diffusions of the form

$$df_t = (e + gf_t)dt + \sigma(f_t)d\tilde{W}_t, \quad \sigma(f_t)^2 = bf_t^2 + cf_t + d.$$

A second necessary condition is that the polynomials are in the domain of the infinitesimal generator, which requires the L^2 integrability of the polynomials; see Hansen, Scheinkman and Touzi (1998) for more details. This is always the case for the square-root and Ornstein-Uhlenbeck processes, but not for the GARCH diffusion model. In the last case, one needs an assumption on the parameters that insure the L^2 integrability of the moments.

Of course, an alternative approach to the methodology of Bollerslev and Zhou (2002) for the identification of the parameter a is the incorporation of the moving-average parameter in the estimation; for instance, by using

$$Cov(RV_t(h), RV_{t-1}(h)) = Cov(IV_t, IV_{t-1}) = a^2 \frac{[1 - \exp(-\lambda)]^2}{\lambda^2}.$$

In their empirical section, Bollerslev and Zhou (2002) used moment conditions fulfilled by the integrated variance process but apply them to the realized variance observations. In

other words, these authors ignore the measurement error $e_t(h)$. However, the conditions (4.1) and (4.2) are the same and, hence, the measurement error problem does not matter for these moment conditions. This feature may explain why the Bollerslev and Zhou (2002) approach works well even if the measurement error problem matters for the moment restrictions on the squared process IV_t^2 .⁹

Instead of adopting a GMM approach, Barndorff-Nielsen and Shephard (2002a) proposed a QML estimation procedure and used the state-space representation of the integrated variance process combined with the Kalman filter.¹⁰ Interestingly, the ARMA representation of the realized variance process that we derived allows us to do the same estimation procedure. More precisely, one can estimate the parameters of the model by minimizing the function

$$\sum_{t=1}^T \frac{(RV_t(h) - m_{t-1}[RV(h)])^2}{2var[\eta_t(h)]} + \frac{1}{2} \log(var[\eta_t(h)]),$$

with $m_0[RV(h)] = E[RV_t(h)] = a_0$. This approach is the same as that of Barndorff-Nielsen and Shephard (2002a) given that $m_{t-1}[RV]$ is the analytical steady-state of the Kalman filter. We leave for future research the use of this estimation method, which is clearly simpler than the methods of Barndorff-Nielsen and Shephard (2002a) and Bollerslev and Zhou (2002).

The previous QML estimator (as well as the GMM one based on (4.2)) depends on h given that the criterion does (with or without leverage effect). Consequently, by considering h as a constant, the asymptotic variance-covariance matrix of the estimator depends on h (but not the limit of the estimator given that it is consistent). Of course, one can also do a double-asymptotic analysis, i.e., by allowing $T \rightarrow +\infty$ and $h \rightarrow 0$, where T is the sample size; this approach is followed by Corradi and Distaso (2002).

5 Empirical Implications

In this section, we derive the empirical implications of the ARMA representations of integrated and realized variances. We consider two examples. The first one is a one-factor affine continuous time stochastic volatility model, i.e.,

$$dx_t = \sigma_t dW_t, \quad d\sigma_t^2 = k(\theta - \sigma_t^2)dt + \sigma\sigma_t dW_{1,t}, \quad (5.1)$$

where W_t and $W_{1,t}$ are independent standard Brownian processes. The numerical results we will provide are based on the parameter estimates reported in Bollerslev and Zhou (2002) obtained by matching the sample moments of the daily realized variances constructed from high-frequency five-minute DM/\$ returns spanning from 1986 through 1996 to the

⁹Recently, Corradi and Distaso (2002) derived sufficient conditions on the speed of convergence of h toward zero (relatively to the sample size T) that allows one to ignore the measurement error in statistical inference.

¹⁰See also Galbraith and Zinde-Walsh (2001) and Maheu and McCurdy (2002) for alternative estimation procedures.

corresponding population moments for the integrated variance. The resulting values are $k = 0.1464$, $\theta = 0.5172$, $\sigma = 0.5789$, implying the existence of a slow mean-reverting factor.

The second example is a two-factor affine continuous time stochastic volatility model, i.e.,

$$dx_t = \sigma_t dW_t, \quad \sigma_t^2 = \sigma_{1,t}^2 + \sigma_{2,t}^2, \quad d\sigma_{i,t}^2 = k_i(\theta_i - \sigma_{i,t}^2)dt + \sigma_i \sigma_{i,t} dW_{i,t}, \quad \text{for } i = 1, 2, \quad (5.2)$$

where W_t , $W_{1,t}$, and $W_{2,t}$ are independent standard Brownian processes. The numerical results we will provide are also based on the parameter estimates reported in Bollerslev and Zhou (2002). The resulting values are $k_1 = 0.5708$, $\theta_1 = 0.3257$, $\sigma_1 = 0.2286$, $k_2 = 0.0757$, $\theta_2 = 0.1786$, $\sigma_2 = 0.1096$, implying the existence of a very volatile first factor, along with a more slowly mean-reverting second factor.

The results concerning the one-factor and two-factor models are reported in Table 1 and Table 2 respectively. In Table 1, we reported for integrated variance (respectively realized variance) the autoregressive and moving-average roots, denoted by γ and λ (respectively $\gamma(h)$ and $\lambda(h)$) and the dynamic coefficients, denoted by α and β (respectively $\alpha(h)$ and $\beta(h)$), of the recursive equation describing the expected value of integrated variance (respectively realized variance), given in (3.19) (respectively (3.23)). We give the same coefficients in Table 2 for the two-factor model. The realized variances are computed by using intra-daily returns at the following lengths: one day, eight hours, four hours, three hours, one hour, thirty minutes, fifteen minutes, ten minutes, five minutes, one minute, and thirty seconds.

Let us start by studying the results of the one-factor model reported in Table 1, which can be summarized as follows. The autoregressive root of integrated variance, IV_t , and realized variance, $RV_t(h)$, coincide for any h , while their moving-average roots are different. However, when h goes to zero, the moving-average root of realized variance denoted by $\lambda(h)$ converges toward the moving-average root of the integrated variance process (denoted by λ). The same convergence result holds for the parameters of the recursive equation of the expected values of integrated and realized variances. As discussed in Section 3, parameter β , defined in the GARCH-like equation describing the expected value of integrated variance, is negative. Therefore, it is also the case for the corresponding coefficient of realized variance (i.e., $\beta(h)$) when h is small. However, when h is not small, the amount of the noise $e_t(h)$ is important and $\beta(h)$ becomes positive. The result in Table 1 indicates that $\beta(h)$ becomes positive for $h \geq 1/288$, which corresponds to realized variance computed with five-minute returns or longer ones. In other words, this is the case in practice, given that empirical papers consider five-minute or thirty-minute returns. Finally, note that the parameters $\alpha(h)$ and $\beta(h)$ for $h = 1$, i.e., squared daily returns coincide with the usual GARCH(1,1) parameters. As shown in the previous section, the ARMA representation of the realized variance $RV_t(1)$ coincides with the (weak) GARCH representation of daily returns.

Consider now the two-factor model. All the results pointed out for the one-factor model are still valid for the two-factor case. However, there is an additional result: there exists a

frequency, which corresponds to a particular value of h , denoted h^* , for which a moving-average root of realized variance equals the non-persistent autoregressive root (i.e., γ_2). From Table 2, it is clear that h^* is between $1/8$ and $1/24$, which correspond to three-hour and one-hour returns respectively. Therefore, $RV_t(h^*)$ is an ARMA(1,1). As a consequence, the structural model, i.e., the two-factor model specified in (5.2), is not completely identifiable but only partially. In other words, one can not estimate all the parameters given in (5.2) by using the ARMA representation of realized variance $RV_t(h^*)$, as did Barndorff-Nielsen and Shephard (2002a) and Bollerslev and Zhou (2002).¹¹

Of course, by taking a different h than h^* , one identifies the model and hence can estimate all the parameters. However, there are two problems with such argument. The frequency h^* is unknown and depends on the parameters that have to be estimated. Therefore, by taking an ad hoc frequency, one is not sure that the model is identified. The second problem is that for h close to h^* , a moving-average root of realized variance is close to the non-persistent autoregressive root. For instance, from Table 2, it is clear that the moving-average root $\lambda_1(h)$ is close to the autoregressive root γ_1 for realized variance computed with intra-daily returns of length between one hour and four hours. Therefore, the model is somewhat weakly identified, a results that may explain why the coefficient k_2 of the persistent factor in Bollerslev and Zhou (2002) was not precisely estimated.¹²

6 Conclusion

This paper has derived the ARMA representation of integrated and realized variances when the spot variance depends linearly on two autoregressive factors, i.e., SR-SARV(2) models. This class of processes includes popular models like affine and positive Ornstein-Uhlenbeck processes, estimated using realized variances respectively by Bollerslev and Zhou (2002) and Barndorff-Nielsen and Shephard (2002a). Such representation is useful for forecasting (see Andersen, Bollerslev and Meddahi, 2002, for an application), and for statistical inference.

Several questions are still open and left for future research. The first one is to study the sign of the expected value of integrated variance. We have shown that the usual sufficient assumption that ensures this positivity is violated and it will therefore be useful to have more insight into this issue. A second interesting question concerns be the study of the potential identification problem that highlighted in the empirical section. A potential solution will be the combination of various realized variances computed with different lengths of intra-daily returns. This approach is currently under investigation.

¹¹Bollerslev and Zhou (2002) consider also moment restriction fulfilled by the square of the integrated variance. Therefore, it is possible that this moment condition identifies the rest of the parameter. Note also that in our analysis, we do not take into account the fact that some moment restrictions used by Bollerslev and Zhou (2002) are valid for integrated variance and not for realized variance.

¹²The standard error reported by Bollerslev and Zhou (2002) for this parameter is 0.8984.

Appendix

Proof of Proposition 3.1. By using (2.7) for one-factor and two-factor models, one gets

$$\begin{aligned}
 E[IV_t \mid f_\tau, p_\tau, \tau \leq t-1] &= \int_{t-1}^t E[\sigma_u^2 \mid f_\tau, p_\tau, \tau \leq t-1] du \\
 &= \int_0^1 E[\sigma_{t-1+u}^2 \mid f_\tau, p_\tau, \tau \leq t-1] du \\
 &= \int_0^1 \left(a_0 + a \exp(-\lambda u) P(f_{t-1}) + \tilde{a} \exp(-\tilde{\lambda} u) \tilde{P}(f_{t-1}) \right) du,
 \end{aligned}$$

i.e.,

$$E[IV_t \mid f_\tau, p_\tau, \tau \leq t-1] = a_0 + a \frac{1 - \exp(-\lambda)}{\lambda} P(f_{t-1}) + \tilde{a} \frac{1 - \exp(-\tilde{\lambda})}{\tilde{\lambda}} \tilde{P}(f_{t-1}). \quad (\text{A.1})$$

When $\tilde{a} = 0$, (A.1) means that

$$IV_t = s_{t-1} + u_t \text{ with } s_{t-1} = a_0 + a \frac{1 - \exp(-\lambda)}{\lambda} P(f_{t-1}) \text{ and } E[u_t \mid f_\tau, p_\tau, \tau \leq t-1] = 0. \quad (\text{A.2})$$

Equation (2.5) implies that

$$P(f_{t-1}) = \exp(-\lambda)P(f_{t-2}) + w_{t-1} \quad \text{with} \quad E[w_{t-1} \mid f_\tau, p_\tau, \tau \leq t-2] = 0.$$

Hence,

$$s_{t-1} = a_0(1 - \exp(-\lambda)) + \exp(-\lambda)s_{t-2} + v_{t-1} \text{ with } v_{t-1} = a \frac{1 - \exp(-\lambda)}{\lambda} w_{t-1}.$$

Therefore, we have

$$E[z_t \mid f_\tau, p_\tau, \tau \leq t-2] = 0 \quad \text{with}$$

$$z_t \equiv IV_t - \exp(-\lambda)IV_{t-1} - a_0(1 - \exp(-\lambda)) = u_t - \exp(-\lambda)u_{t-1} + v_{t-1}, \quad (\text{A.3})$$

i.e., z_t is a MA(1) process given that its variance is finite. Thus, there exists a white noise η_t and a real number β , with $|\beta| < 1$, such that $z_t = \eta_t - \beta\eta_{t-1}$,

$$-\frac{\beta}{1 + \beta^2} = \frac{Cov(z_t, z_{t-1})}{Var[z_t]} \equiv \rho \quad \text{and} \quad Var[\eta_t] = \frac{Var[z_t]}{1 + \beta^2} = \frac{D_0}{1 + \beta^2}.$$

The equation $\beta + \rho(1 + \beta^2) = 0$ admits one unique solution β such that $|\beta| < 1$ and is given in (3.18). To achieve the proof, we need $Var[z_t]$ and $Cov(z_t, z_{t-1})$. Equation (A.3) implies that

$$Var[z_t] = (1 + \gamma^2)Var[IV_t] - 2\gamma Cov[IV_t, IV_{t-1}] = D_{10}(\gamma, Var[IV_t], Cov[IV_t, IV_{t-1}]),$$

with $\gamma = \exp(-\lambda)$, and

$$Cov(z_t, z_{t-1}) = -\gamma Var[IV_t] + (1 + \gamma^2)Cov[IV_t, IV_{t-1}] - \gamma Cov[IV_t, IV_{t-2}].$$

But IV_t is an ARMA(1,1) where the autoregressive parameter equals γ . Therefore

$$Cov[IV_t, IV_{t-2}] = \gamma Cov[IV_t, IV_{t-1}].$$

Thus,

$$Cov(z_t, z_{t-1}) = -\gamma Var[IV_t] + Cov[IV_t, IV_{t-1}] = D_{11}(\gamma, Var[IV_t], Cov[IV_t, IV_{t-1}]).$$

Finally,

$$m_{t-1}[IV] = IV_t - \eta_t = (1 - \gamma)a_0 + \gamma IV_{t-1} - \beta \eta_{t-1} = \omega + \gamma IV_{t-1} - \beta(IV_{t-1} - m_{t-2}[IV]),$$

i.e., (3.19). ■

Proof of Proposition 3.2. By combining (3.3) and (A.2), one gets

$$RV_t(h) = s_{t-1} + u_t(h) \text{ with } u_t(h) = u_t + e_t(h), \text{ and } E[u_t(h) \mid f_\tau, p_\tau, \tau \leq t-1] = 0,$$

given that Meddahi (2002b) shows $E[e_t(h) \mid f_\tau, p_\tau, \tau \leq t-1] = 0$. The rest of the proof is exactly the same as for the proof of Proposition 3.1; in particular, we have

$$Var[\eta_t(h)] = \frac{D_0(h)}{1 + \beta(h)^2}. \blacksquare$$

Proof of Proposition 3.3. Equation (A.1) implies that IV_t is an ARMA(2,2) with autoregressive roots that equal $\exp(-\lambda)$ and $\exp(-\tilde{\lambda})$. The rest of the proof is an application of Proposition 2.1 and Proposition 2.3 of Meddahi (2002b); in particular,

$$Var[\eta_t] = \frac{D_0}{1 + \beta_1^2 + \beta_2^2}.$$

Finally, the proof of (3.26) is similar to the proof of (3.19). ■

Proof of Proposition 3.4. By combining (3.3) and (A.1), one gets

$$E[RV_t(h) \mid f_\tau, p_\tau, \tau \leq t-1] = a_0 + a \frac{1 - \exp(-\lambda)}{\lambda} P(f_{t-1}) + \tilde{a} \frac{1 - \exp(-\tilde{\lambda})}{\tilde{\lambda}} \tilde{P}(f_{t-1}), \quad (\text{A.4})$$

given that Meddahi (2002b) shows $E[e_t(h) \mid f_\tau, p_\tau, \tau \leq t-1] = 0$. Equation (A.4) implies that $RV_t(h)$ is an ARMA(2,2) with autoregressive roots that equal $\exp(-\lambda)$ and $\exp(-\tilde{\lambda})$. The rest of the proof is similar to the proof of Proposition 3.3. In particular,

$$Var[\eta_t(h)] = \frac{D_0(h)}{1 + \beta_1(h)^2 + \beta_2(h)^2}. \blacksquare$$

Table 1: ARMA(1,1) representation of daily integrated and realized variances for the one-factor model

IV			γ	λ	α	β
			.966	-.268	1.23	-.268
RV(h)	1/h	Freq	$\gamma(h)$	$\lambda(h)$	$\alpha(h)$	$\beta(h)$
	1	1 day	.966	.898	.0678	.898
	3	8 hours	.966	.835	.130	.835
	6	4 hours	.966	.777	.189	.777
	8	3 hours	.966	.747	.218	.747
	24	1 hour	.966	.603	.362	.603
	48	30 mn	.966	.486	.480	.486
	96	15 mn	.966	.352	.614	.352
	144	10 mn	.966	.269	.697	.269
	288	5 mn	.966	.128	.838	.128
	1440	1 mn	.966	-.126	1.09	-.126
	2880	30 sec	.966	-.188	1.15	-.188

Table 2: ARMA(2,2) representation of daily integrated and realized variances for the two-factor model

IV		γ_1	γ_2	λ_1	λ_2	α_1	α_2	β_1	β_2	
		.565	.927	-.306	.858	.940	-.739	.552	.215	
RV(h)	1/h	Freq	$\gamma_1(h)$	$\gamma_2(h)$	$\lambda_1(h)$	$\lambda_2(h)$	$\alpha_1(h)$	$\alpha_2(h)$	$\beta_1(h)$	$\beta_2(h)$
	1	1 day	.565	.927	.665	.793	.0337	-.0242	1.46	-.500
	3	8 hours	.565	.927	.638	.771	.0837	-.0611	1.41	-.463
	6	4 hours	.565	.927	.603	.750	.139	-.103	1.35	-.421
	8	3 hours	.565	.927	.583	.741	.169	-.126	1.32	-.398
	24	1 hour	.565	.927	.460	.711	.321	-.244	1.17	-.280
	48	30 mn	.565	.927	.342	.707	.442	-.340	1.05	-.184
	96	15 mn	.565	.927	.201	.721	.570	-.441	.922	-.084
	144	10 mn	.565	.927	.115	.736	.641	-.498	.851	-.026
	288	5 mn	.565	.927	-.024	.769	.747	-.583	.745	.059
	1440	1 mn	.565	.927	-.226	.830	.888	-.697	.604	.173
	2880	30 sec	.565	.927	-.264	.843	.913	-.717	.580	.193

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